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AN

ELEMENTARY TREATISE

ON

MENSURATION

BY

B. T. MOORE, M.A.

FELLOW OF PEMBROKE COLLEGE, CAMBRIDGE, AND
LATE JUNIOR PROFESSOR OF MATHEMATICS IN THE STAFF COLLEGE.

With Numerous Examples.

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PREFACE.

The following Treatise differs essentially from many existing works on Mensuration, inasmuch as all the rules and formulæ employed in it are strictly demonstrated; and, in no case, is any one of them introduced to the notice of the Student in the repulsive form of an empirical rule. At the same time, the methods employed are such that the Student who takes up this Treatise will not require any higher knowledge than a fair acquaintance with Euclid, Algebra, and Plane Trigonometry, together with a knowledge of the *forms* of the Conic Sections, and of their more simple properties. The want of a Treatise of this nature has long been felt and admitted, and it is in the hope of supplying this want that I have undertaken the present work.

The arrangement of the subject, and its scope, will be seen from the Table of Contents. The first five Chapters treat, chiefly, of the areas of plane rectilinear figures, and of the volumes of solids bounded by them. In Chapter II. I have given the Method of Coordinates as applied to the determination of the areas of plane rectilinear figures. This method, on account of its simplicity, generality, and facility of application, is well worthy of notice, and only requires to be known, to be generally employed.

In Chapter III. will be found the two common rules for approximating to the areas of plane figures bounded by irregular or free curves; and I have introduced these rules thus early in the Treatise, because they are easily applied, and complete the subject of the areas of irregular plane figures; but their full investigation must be deferred to a later Chapter.

In Chapter V. Art. 46, I have given a new and very simple proof of the formula for the volume of a prismoid, or, as it is called, the "Prismoidal Formula." The solid employed in the proof is a four-sided prismoid. The formula might be proved directly, by the same method, for a prismoid of any number of sides; but the investigation would be less simple, and nothing would be gained in generality, since it is easy to generalise the formula for all prismoidal solids as is done in Art. 49.

Chapter VI. treats of the Method of Limits,—a method of which it is sufficient to say that it is *indispensable* to the Student who wishes to *understand* the mensuration of curve lines and surfaces, and of the areas and volumes which they respectively enclose.

In Chapter IX. I have extended the Prismoidal Formula to a great number of solids bounded by curve surfaces; and have shown how to determine, in any given case, whether the formula is applicable or not.

In the last Chapter will be found a general investigation of the methods of approximating to the areas of figures bounded by free curves, and to the volumes of solids bounded by free surfaces. In this Chapter, in addition to the two well-known rules given in Chapter III., will be found a remarkable and very simple formula due to the late Mr Weddle, and which I have introduced under the name of "Weddle's Rule." This formula comes into use most conveniently in cases in which Simson's Rule would not apply, or would not give so close an approximation; and a modification of the same formula, to render it applicable to volumes, forms an equally useful addition to the Prismoidal Formula.

The Student will find a considerable number of Examples and Problems for practice appended to the Treatise, some of which are worked out at length: for any knowledge which he may require respecting conventional measures, he is referred to Mr Wrigley's Arithmetic.

B. T. M.

PEMBROKE COLLEGE, CAMBRIDGE,
April 25, 1863.

TABLE OF CONTENTS.

CHAPTER I.

Areas of Plane Rectilinear Figures.

ARTICLE PAGE

1. OBJECTS of Mensuration 1
2. Units of area 2
3. Area of a rectangle; first proof. 2
4. Area of a rectangle; second proof. 5
5. Area of a parallelogram........ 7
6. Area of a triangle 8
7. Area of a quadrilateral 9
8. Area of any rectilinear figure 12
9. Area of a trapezoid 12
10. Area of a quadrilateral inscribed on a circle 13
11. Area of a regular polygon........ 17
12. Area of the surface of a right prism. 19
13. Area of the surface of an oblique prism 20
14. Area of the surface of a frustum of a prism 21
15. Area of the surface of a regular pyramid 22
16. Area of the surface of a frustum of a regular pyramid 23

CHAPTER II.

Method of Coordinates, and its application to the determination of the Areas of Plane Figures.

ARTICLE PAGE

17. General explanation 25
18. Distance between two points in terms of their coordinates 28
19. Area of a triangle in terms of the coordinates of its angular points 29
20. Area of any rectilinear figure in terms of the coordinates of its angular points 31
21. Method of measuring coordinates when the points are inaccessible 34
22. Method of measuring coordinates when the points are accessible; advantage of this method 35
23. Example in coordinates,—triangle 36
24. Example in coordinates,—polygon 37

CHAPTER III.

Approximation to the Areas of Plane Curvilinear Figures.

25. First method of approximation 40
26. Second method of approximation, or "Simson's Rule" 41
27. Area of any curvilinear figure 43
28. Remarks on the preceding methods........ 44

CHAPTER IV.

Areas of the sections of certain Military Earthworks.

29. Area of the transverse section of a parapet........ 45
30. The same, by reduction to a triangle 46
31. Area of the transverse section of a ditch. 47

ARTICLE PAGE
32. To find the breadth or depth of a ditch under certain conditions ... 49
33. Modification of the preceding for expansion of earth 51

CHAPTER V.

Volumes of Solids contained by Plane Rectilinear Figures.

34. Units of volume ... 53
35. Volume of a rectangular parallelepiped; first proof. 53
36. Volume of a rectangular parallelepiped; second proof. ... 57
37. Volume of any parallelepiped ... 58
38. Volume of a prism ... 60
39. Volume of a pyramid ... 63
40. Volume of a frustum of a pyramid ... 67
41. Volume of a frustum of a triangular prism ... 69
42. Volume of a frustum of a regular prism ... 71
43. The same by another method ... 73
44. Volume of a frustum of an irregular prism ... 74
45. Volume of any portion of a parapet ... 76
46. Volume of a prismoid ... 77
47. Volume of a triangular prismoid ... 80
48. Volume of a frustum of a pyramid, deduced from the prismoid ... 81
49. Generalization of prismoidal solids ... 81
50. Volume of a railway-cutting or embankment ... 82
51. The same when on a curve ... 84

CHAPTER VI.

On the Method of Limits.

52. Definition of a limit: illustrations ... 85
53. Definition of a curve: illustrations ... 86
54. Definition of the tangent to a curve. ... 87
55. Newton's first Lemma ... 88
56. Newton's second Lemma ... 89

ARTICLE PAGE
57. Newton's third Lemma 90
58. Application of Lemmas II, III, to solids of revolution 90
59. Newton's fourth Lemma 91
60. Definition of similar curvilinear figures 93
61. Newton's fifth Lemma 93
62. Test of the similarity of curvilinear figures 93
63. Newton's sixth Lemma 94
64. A property of similar curves 95
65. Newton's seventh Lemma 96
66. Limiting values of $\frac{\sin\theta}{\theta}$ and $\frac{\tan\theta}{\theta}$ 97
67. Limiting value of $\frac{1^p+2^p+3^p+\ldots\ldots+n^p}{n^{p+1}}$ 99

CHAPTER VII.

Areas of Plane Curvilinear Figures.

68. Objects of the Chapter 103
69. Area and circumference of a circle 103
70. Remark on the investigation of the circumference of a circle: determination of π 105
71. Areas of a sector and segment of a circle 107
72. Area of a circular annulus 110
73. Area of a lune 111
74. Area and circumference of an oval 112
75. Areas of an ellipse and its segments 114
76. Area of a segment of a parabola 116
77. Length and area of the involute of a regular polygon 118
78. Length and area of the involute of any polygon ... 122
79. Length and area of the involute of a circle 123
80. Length and area of a cycloid 128

CHAPTER VIII.

Areas and Volumes of Solids bounded by Curve Surfaces.

ARTICLE PAGE
81. Area of the surface, and volume, of a circular cylinder ... 132
82. Area of the surface, and volume, of any cylinder ... 133
83. Volume of any frustum of a circular cylinder ... 134
84. Area of the surface, and the volume, of a circular cone ... 134
85. Volume of any cone ... 135
86. Surface of a frustum of a circular cone ... 136
87. Volume of a frustum of a circular cone ... 137
88. Volume of a frustum of any cone ... 138
89. Formula of Art. 87 deduced from that of Art. 40... 139
90. Volume of a segment of a paraboloid of revolution. 139
91. Volume of a frustum of a paraboloid ... 140
92. Surfaces of a sphere, its segments, and zones ... 141
93. Volumes of a sphere, its segments, and zones ... 143
94. Volume of a spherical shell ... 145
95. Volume of a sphere, deduced from that of a cone... 146
96. Volumes of the prolate and oblate spheroids ... 147
97. Volume of an ellipsoid ... 148
98. Volume and surface of a circular ring, and of certain other solids ... 150
99. Volume of a spiral spring, with example ... 153

CHAPTER IX.

The "Prismoidal Formula," and its application.

100. General proof of the Prismoidal Formula ... 156
101. Discrimination of prismoidal solids: solids of revolution ... 156
102. Discrimination of prismoidal solids: solids not of revolution ... 163
103. General test of prismoidal solids ... 166
104. Examples of the application of the Prismoidal Formula ... 167

CHAPTER X.

On the Parabolic Spindle; and on Cask gauging.

ARTICLE PAGE

105. Definition of a spindle: varieties of spindles......... 171
106. Volumes of a parabolic spindle, and its frusta 171
107. Volumes of solids employed in cask gauging 174
108. Order of magnitude of the preceding solids 176
109. Investigation of Hutton's formula 177
110. Method of gauging by four dimensions 180
111. Limits of the errors made by ordinary rules 180
112. Examples of cask gauging................................. 181
113. Adaptation of formulæ to practice 183

CHAPTER XI.

Approximation to the Areas of Figures bounded by Free Curves and the Volumes of Solids bounded by Free Surfaces.

114. Objects of the Chapter: statement of method employed 184
115. General explanation of the method of approximation 185
116. General expression for the area of an approximate curvilinear figure....................................... 186
117. Degrees of approximation; and the resulting formulæ 187
118. Proof of "Weddle's Rule"................................. 189
119. Generalization of the preceding formulæ 192
120. Recapitulation of the general formulæ; and remarks 195
121. Examples of approximation to areas 196
122. Extension of the preceding formulæ to the volumes of solids bounded by free surfaces, with example 201

A Collection of Examples and Problems.

PAGE
Examples and Problems on Chapter I. 204
" " " " II. 216
" " " " III. 218
" " " " IV. 223
" " " " V. 227
" " " " VII. 241
" " " " VIII. 257
" " " " IX. 273
" " " " X. 276
" " " " XI. 277

ERRATA.

Page 24, line 1, *for* "frustrum" *read* "frustum."
... 36, ... 18, *for* "14939·15" *read* "15129·15."
... 197, ... 4, *for* "a point" *read* "a point in the curve."

MENSURATION.

◆

CHAPTER I.

AREAS OF PLANE RECTILINEAR FIGURES.

1. THE object of that branch of mathematical science which is called Mensuration, is the investigation of the lengths of lines, the areas of surfaces, and the volumes of solids, whatever may be their forms. These investigations cannot be made, in all cases, without the aid of the Differential and Integral Calculus; but, for a great number of cases, and these by far the most important in their practical applications, much simpler methods will suffice. We suppose the student already acquainted with the determination of the lengths of straight lines from various data, as this forms an important part of every treatise on Plane Trigonometry. We shall therefore confine our attention to the lengths of curved lines, the areas of surfaces, and the volumes of solids, so far as these can be conveniently investigated without having recourse to the Differential and Integral Calculus.

2. Now in measuring the length of a line, whether it be a straight line or a curve, the problem, in every case, is to determine the ratio which the given line bears to some determinate line which is chosen arbitrarily, and is called the unit of length.

In the same manner, in measuring the area of a surface, whether plane or curved, the problem is to determine the ratio which the given surface bears to some determinate surface which is chosen arbitrarily, and is called the unit of area.

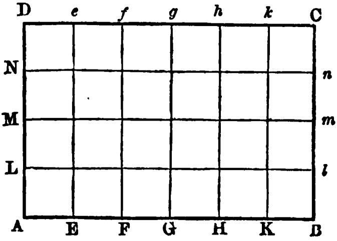
Definition of a Unit of Area. The area contained by two straight lines at right angles, each of which is a unit of length, is called a unit of area.

In other words the unit of area is a square, a side of which is a unit of length.

If the unit of length be an inch, the unit of area will be called a square inch; and, if the unit be a foot, a yard, or a mile, the unit of area will be called a square foot, a square yard, or a square mile respectively.

We shall, in general, represent the number of units of area in any surface by capital letters, as A, and the number of units of length in lines by small letters, as a.

3. *To find an expression for the area of a rectangle in terms of two adjacent sides.*



Let $ABCD$ be a rectangle,

a the number of units of length in AB,
b .... AD,
A............................... area in $ABCD$;

we shall prove that

$$A = ab.$$

There will be three cases:—

(1) When a and b are *integers.*

(2) When a and b are, either or both of them, *fractional.*

(3) When a and b are *incommensurable.*

Case (1). When a and b are *integers.*

Divide AB into a equal parts in the points E, F, G, H, K,

and AD...... b.............................. L, M, N;

and through these points of division draw the straight lines Ee, Ff, Gg, Hh, Kk parallel to AD, and Ll, Mm, Nn parallel to AB.

These lines divide the rectangle into a number of squares, each of which is a unit of area, and it will be evident that the number of these squares in *each* of the rectangles Al, Lm, Mn, NC will be a, and therefore the number of these squares, or units of area, in the rectangle $ABCD$ will be a taken b times, or ab.

$$\therefore A = ab.$$

Obs. In the figure, AB is divided into *six* equal parts and AD into *four* equal parts. This is merely for convenience, because we cannot divide AB into a equal parts until we know what a is, but we can *suppose* AB and AD divided into a and b equal parts respectively, and reason upon them thus supposed to be divided.

Case (2). When a and b are, either or both of them, *fractional.*

We shall suppose them both fractional.

Let α represent the unit of length, so that

$$AB = a \,.\, \alpha, \qquad AD = b \,.\, \alpha,$$

and suppose $$a = p + \frac{h}{m}, \qquad b = q + \frac{k}{n},$$

where p and q are positive integers, and $\frac{h}{m}$ and $\frac{k}{n}$ proper fractions in their lowest terms.

Now take another unit of length β, less than α, and such that

$$\alpha = mn\beta.$$

Then
$$a.\alpha = \left(p + \frac{h}{m}\right) mn\beta = (pm + h)\, n\beta,$$

$$b.\alpha = \left(q + \frac{k}{n}\right) mn\beta = (qn + k)\, m\beta\,;$$

and these latter expressions for $a.\alpha$ and $b.\alpha$ being *whole numbers* of the unit of length β, their product will, by case (1), be the number of units of area in the rectangle, each of which is the square upon the unit of length β, and will be represented by β^2.

$$\therefore \text{ Area} = (pm + h)\,(qn + k)\, mn \text{ units of area } \beta^2,$$

$$= \frac{pm + h}{m} \cdot \frac{qn + k}{n}\, m^2n^2 \ldots\ldots\ldots\ldots\ldots\ldots,$$

$$= \left(p + \frac{h}{m}\right)\left(q + \frac{k}{n}\right) \text{ units of area } \alpha^2,$$

because a unit of area corresponding to the unit of length α contains m^2n^2 units of area corresponding to the unit of length β.

Hence, as in the first case,

$$A = ab.$$

Case (3). When a and b are *incommensurable.*

In this case, although we cannot find any unit which will measure AB and AD, yet, as we may take a unit *as small as we please*, we can always find two lines *which shall be commensurable*, and which shall differ from AB and AD by quantities *less than any that can be assigned;* and their product will be the area of a rectangle differing from the given rectangle by a quantity *less than any assignable.*

Hence, in this case also, the number of units of area in the rectangle must be the product of the numbers of units of length in two of its adjacent sides.

Hence, in all cases, we have the formula

$$A = ab.$$

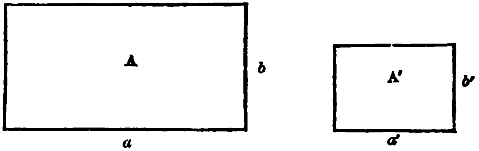
Obs. In this treatise we shall, for the sake of brevity, speak of the product of one line by another; in all such cases it is to be understood that we mean the number of units of length in one line multiplied by the number of units of length in the other.

We shall now give another and independent proof of the preceding proposition derived from pure Geometry, both on account of its general elegance and completeness, and also to exhibit the subject from a different point of view.

4. *To find an expression for the area of a rectangle.*

In Euclid's *Elements of Geometry*, Book VI. Proposition 23, it is proved that "Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides;" and therefore rectangles, which are equiangular parallelograms, have to one another this same ratio.

Let A and A' represent the surfaces, or areas, of two rectangles, and let their sides be represented by a, b, and a', b', respectively.



Then, by the above-mentioned proposition,

$$\frac{A}{A'} = \frac{a}{a'} \times \frac{b}{b'} = \frac{ab}{a'b'},$$

$$\therefore A = A' \frac{ab}{a'b'} \quad \ldots\ldots. \quad (1).$$

This equation is the foundation of the measurement of surfaces, but before we can apply it in practice we must define some standard of measurement, or, in other words, we must select some particular area for a unit by which all other areas shall be measured. The problem of finding an area will then be reduced to that of finding the ratio between the area of any given figure and that of the particular area chosen for a unit.

The usual convention with regard to units of area is that the area, or surface, which is contained by two units of length at right angles to each other, shall be called a unit of area. Thus, if we take an inch for the unit of length, the area contained by two inches at right angles to each other will be the unit of area, and is usually called a square inch.

Similarly, if we take a foot, a yard, or a mile for the unit of length, the corresponding units of area will be those areas contained by two feet, two yards, or two miles, at right angles to each other, and will be called respectively, a square foot, a square yard, and a square mile.

Now if in the right-hand figure we suppose a' and b' to be, each of them, a unit of length, A' will become a unit of area, and equation (1) will become

$$A = ab \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This equation shows that the number of units of area in a rectangle is that number which is the product of the numbers of units of length in two adjacent sides.

It may be asked why we choose the particular areas as already defined for units of measurement. The answer is, that equation (2) would not be under so simple a form were any other area taken for a unit. If, for example, we take that area for a unit which is contained by 2 units of length on one side and 3 units of length on an adjacent side, then in equation (1) we shall have

$$A' = 1 \textit{ when } a' = 3 \textit{ and } b' = 2,$$

and the resulting equation will be

$$A = \frac{ab}{3 \times 2},$$

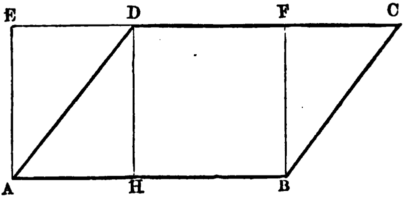
which is less simple in form than equation (2); and the same objection will apply to any other unit except that which we have chosen.

The proof just given includes every case which can occur, whether the sides a and b be represented by whole numbers of the unit of length, or by fractional or incommensurable numbers, since it is derived from pure geometry, which is independent of all units of measurement.

5. *To find an expression for the area of a parallelogram—*

(1) In terms of one side and the perpendicular distance between it and the opposite side; commonly called base and perpendicular.

(2) In terms of two adjacent sides and the angle included between them.



Case (1). *Area in terms of one side and the perpendicular distance between it and the opposite side.*

Let $ABCD$ be a parallelogram, and complete the rectangle $ABFE$. Then $ABCD$, $ABFE$ are parallelograms upon the same base and between the same parallels, and are therefore equal to each other. Euclid, I. 35.

But, area of rectangle $ABFE = AB \times BF$, Arts. 3 or 4.

$\therefore$ also area of parallelogram $ABCD = AB \times BF$.

Hence if c = number of units of length in AB,

p = ... BF,

A = ... area in the parallelogram,

$$A = pc,$$

$$\text{and } \therefore \log A = \log p + \log c.$$

Case (2). ***Area in terms of two adjacent sides and the angle included between them.***

In the figure draw DH perpendicular to AB,

$$\therefore BF = DH = AD \sin BAD,$$

$$\therefore \text{area} = AB \times AD \sin BAD;$$

or, if b = number of units of length in AD,

θ = number of angular units in BAD,

$$A = bc \sin \theta,$$

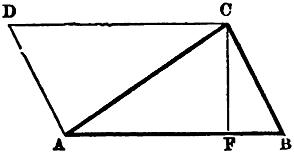
$$\text{and } \log A = \log b + \log c + \log \sin \theta.$$

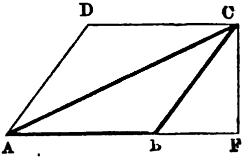
6. *To find an expression for the area of a triangle*—

(1) In terms of one side and the perpendicular upon it from the opposite angular point; commonly called base and perpendicular.

(2) In terms of two sides and the included angle.

(3) In terms of the three sides.





Case (1). ***Area in terms of one side and the perpendicular upon it from the opposite angular point.***

Let ABC be a triangle. Complete the parallelogram $ABCD$, and draw CF perpendicular to AB, or AB produced, as the case may require.

Then the parallelogram is double of the triangle. Euclid, I. 41.

Therefore area of triangle $ABC = \frac{1}{2}$ area of parallelogram $ABCD$,

$$= \tfrac{1}{2} AB \times CF. \text{ Art. 5.}$$

Hence if c = number of units of length in AB,

p = ... CF,

and A = ... area in ABC,

$$A=\tfrac{1}{2}pc,$$

$$\text{and }\log A=\log p+\log c-\log 2.$$

Case (2). *Area in terms of two sides and the included angle.*

From the preceding case we have

$$\text{Area}=\tfrac{1}{2}pc,$$

$$\text{and }p=CF=AC\sin BAC=b\sin A,$$

according to the usual notation in Trigonometry, in which a, b, c represent the sides respectively opposite to the angles A, B, C.

$$\therefore\ \text{Area}=\tfrac{1}{2}bc\sin A,$$

$$\text{and }\log\text{area}=\log b+\log c+\log\sin A-\log 2.$$

Case (3). *Area in terms of the three sides.*

From the last case we have

$$\text{Area}=\tfrac{1}{2}bc\sin A\,;$$

and by Trigonometry we know that

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)},$$

$$\text{where}\quad 2s=a+b+c,$$

Substituting this value of sin A in the expression for the area, we have

$$\text{Area}=\sqrt{s(s-a)(s-b)(s-c)},$$

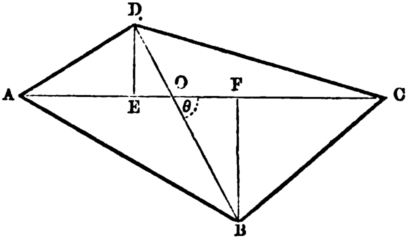
$$\text{and}\quad 2\log\text{area}=\log s+\log(s-a)+\log(s-b)+\log(s-c).$$

7. *To find an expression for the area of a quadrilateral figure—*

(1) In terms of a diagonal and the perpendiculars upon it from the opposite angular points.

(2) In terms of the diagonals and the angle included between them.

(3) **In terms of the four sides and two opposite angles.**



Case (1). *Area in terms of a diagonal and the perpendiculars upon it from the opposite angular points.*

Let $ABCD$ be a quadrilateral figure.

Draw the diagonal AC, and BF, DE perpendicular to AC.

Let d = number of units of length in AC,
p_1 = ... BF,
p_2 = ... DE,
A = ... area in the quadrilateral.

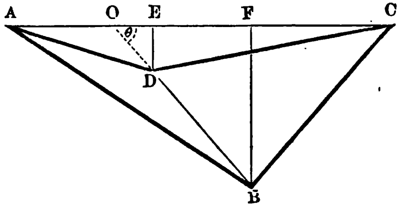
Now A = area of $\triangle ABC$ + area of $\triangle ADC$.

$$\therefore\ A = \tfrac{1}{2}AC \times BF + \tfrac{1}{2}AC \times DE,$$

$$= \tfrac{1}{2}AC(BF + DE),$$

$$= d\frac{p_1 + p_2}{2}.$$

If the diagonal AC fall *without* the quadrilateral as represented in the following figure;



we shall evidently have

$$\begin{aligned} A &= \text{area of } \triangle ABC - \text{area of } \triangle ADC \\ &= \tfrac{1}{2} AC \times BF - \tfrac{1}{2} AC \times DE \\ &= \tfrac{1}{2} AC(BF - DE) \\ &= d\,\frac{p_1 - p_2}{2}. \end{aligned}$$

Case (2). *Area in terms of the diagonals and the angle included between them.*

In each figure join BD, and in the latter produce BD to meet AC in O.

Let $AC = d_1$, $BD = d_2$, and the angle $BOC = \theta$.

Also let $OB = x$, $OD = y$.

Then, in the former figure,

$$\begin{aligned} A &= \triangle ABC + \triangle ADC \\ &= \tfrac{1}{2} p_1 d_1 + \tfrac{1}{2} p_2 d_1 \\ &= \tfrac{1}{2} d_1 (p_1 + p_2) \\ &= \tfrac{1}{2} d_1 (x \sin\theta + y \sin\theta) \\ &= \tfrac{1}{2} d_1 (x + y) \sin\theta \\ &= \tfrac{1}{2} d_1 d_2 \sin\theta; \end{aligned}$$

and in the latter figure,

$$\begin{aligned} A &= \triangle ABC - \triangle ADC \\ &= \tfrac{1}{2} p_1 d_1 - \tfrac{1}{2} p_2 d_1 \\ &= \tfrac{1}{2} d_1 (p_1 - p_2) \\ &= \tfrac{1}{2} d_1 (x \sin\theta - y \sin\theta) \\ &= \tfrac{1}{2} d_1 (x - y) \sin\theta \\ &= \tfrac{1}{2} d_1 d_2 \sin\theta; \end{aligned}$$

∴ for both figures,

$$A = \tfrac{1}{2} d_1 d_2 \sin\theta,$$

and $$\log A = \log d_1 + \log d_2 + \log \sin\theta - \log 2.$$

These formulæ are particularly useful in the survey of an estate or other enclosure. The four points A, B, C, D

should be chosen so as to include as much as possible of the estate. The diagonals and the angle between them, the perpendiculars and the four sides should all be measured, and the area calculated by each of the above formulæ.

A comparison of the results will test the accuracy with which the measurements have been made.

Also any side, as BC, may be calculated from the measured lines OB, OC, and the measured angle BOC, and the result compared with the *measured* length of BC.

In this way the fundamental lines of the survey, viz., the four sides and the two diagonals, should all be verified before the filling in of the work is commenced.

Case (3). *Area in terms of the four sides and two opposite angles.*

Let the sides BC, CD, DA, AB, taken in order, be represented by a, b, c, d, respectively, and let ABC and CDA be the given angles.

Then, in the first figure,

$$A = \tfrac{1}{2}\, ad \sin B + \tfrac{1}{2}\, bc \sin D,$$

and in the second,

$$A = \tfrac{1}{2}\, ad \sin B - \tfrac{1}{2}\, bc \sin (360^\circ - D),$$

$$\text{but } \sin (360^\circ - D) = -\sin D;$$

$$\therefore\ A = \tfrac{1}{2}\, ad \sin B + \tfrac{1}{2}\, bc \sin D,$$

and the same formula is true for both cases.

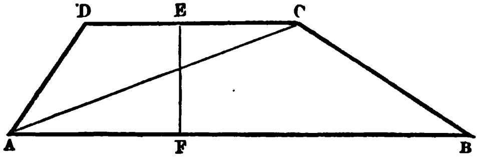
8. We may now find the area of any rectilinear figure, for every rectilinear figure can be divided into a number of triangles, and the areas of these triangles may be found by the formulæ already given, and their sum will be the area of the rectilinear figure; but more convenient methods will be given in a subsequent chapter.

9. *To find an expression for the area of a trapezoid.*

Def. A trapezoid is a four-sided figure of which two opposite sides are parallel.

We shall suppose the parallel sides and the perpendicular distance between them to be given.

Let $ABCD$ be a trapezoid, the sides AB and CD being parallel to each other.



Draw EF perpendicular to AB or CD, and join AC.

Let $AB=a$, $CD=b$, $EF=p$, and $A=$ area of the figure.

$$\therefore\ A = \text{area of } \Delta ABC + \text{area of } \Delta ADC$$

$$= \frac{AB \times EF}{2} + \frac{DC \times EF}{2} \qquad \text{Art. 6. (1).}$$

$$= \frac{AB+CD}{2} \times EF;$$

$$\therefore\ A = \frac{a+b}{2}\,p,$$

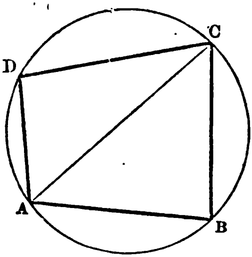
$$\text{and}\quad \log A = \log(a+b) + \log p - \log 2.$$

This proposition will, of course, include the case in which one of the sides AD or BC is itself perpendicular to the two parallel sides. We shall find this latter case of great use in the application of the method of coordinates to the determination of areas.

10. *To find an expression for the area of a quadrilateral figure inscribed in a circle—*

(1) In terms of the sides and one angle.

(2) In terms of the sides only.



Case (1). Let $ABCD$ be a quadrilateral figure inscribed in a circle. Join AC.

Let $BC=a$, *one* of the sides opposite the angle A,

$CD=b$, $DA=c$, $AB=d$,

and let $A=$ area of the quadrilateral.

By Euclid, III. 22, the angles B and D are together equal to two right angles.

Now, area of quadrilateral = area of triangle ABC

+ area of triangle ACD;

$$\therefore A=\frac{ad}{2}\sin B+\frac{bc}{2}\sin D.$$

But $B+D=\pi$, $\therefore D=\pi-B$, $\therefore \sin D=\sin B$.

Hence $$A=\frac{ad+bc}{2}\sin B.$$

Thus we know the area when the four sides and any one angle are given.

Case (2). We now proceed to find the area when the four sides only are known.

To do this we must eliminate $\sin B$ from the expression

$$\frac{ad+bc}{2}\sin B.$$

Let $AC=x$, then by Trigonometry we have

$$x^2=b^2+c^2-2bc\cos D,$$

from the triangle ACD,

but $D=\pi-B,\qquad \therefore \cos D=-\cos B,$

$$\therefore x^2=b^2+c^2+2bc\cos B,$$

also from the triangle ABC,

$$x^2=a^2+d^2-2ad\cos B;$$

$\therefore$ by subtraction,

$$0=b^2-a^2+c^2-d^2+2(bc+ad)\cos B,$$

$$\therefore \cos B=\frac{a^2-b^2+d^2-c^2}{2(ad+bc)}.$$

Again, $\sin^2 B=1-\cos^2 B=(1+\cos B)(1-\cos B)$.

Now $\quad 1+\cos B=1+\dfrac{a^2-b^2+d^2-c^2}{2(ad+bc)}$

$$=\frac{(a^2+d^2+2ad)-(b^2+c^2-2bc)}{2(ad+bc)}$$

$$=\frac{(a+d)^2-(b-c)^2}{2(ad+bc)}$$

$$=\frac{(a+d+b-c)(a+d+c-b)}{2(ad+bc)},$$

and $\quad 1-\cos B=1-\dfrac{a^2-b^2+d^2-c^2}{2(ad+bc)}$

$$=\frac{(b^2+c^2+2bc)-(a^2+d^2-2ad)}{2(ad+bc)}$$

$$=\frac{(b+c)^2-(a-d)^2}{2(ad+bc)}$$

$$=\frac{(b+c+a-d)(b+c+d-a)}{2(ad+bc)}.$$

Substituting in the expression for $\sin^2 B$ these values of $1+\cos B$ and $1-\cos B$, we obtain

$$\sin^2 B=\frac{(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)}{4(ad+bc)^2}.$$

Now let $$2s=a+b+c+d;$$

$$\therefore\ 2(s-a)=b+c+d-a,$$
$$2(s-b)=a+c+d-b,$$
$$2(s-c)=a+b+d-c,$$
$$2(s-d)=a+b+c-d.$$

Substituting these values in the expression for $\sin^2 B$, and extracting the square root, we have

$$\sin B=\pm\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ad+bc)},$$

and we take the positive sign because the angle B must be less than 2 right angles,

$$\therefore\ \frac{ad+bc}{2}\sin B=\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

But the expression on the left-hand side is that which we have already found for the area, hence also

$$A=\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

an expression for the area in terms of the sides.

This last expression has the advantage of being adapted to logarithmic computation, and of being independent of the angles.

We obtain from it

$$2\log A=\log(s-a)+\log(s-b)+\log(s-c)+\log(s-d).$$

Cor. If we suppose the quadrilateral to become a triangle by the vanishing of any one side, say the side AB,

then $d=0$, and $2s=a+b+c$,

and the expression for its area will be

$$\sqrt{(s-a)(s-b)(s-c)s}.$$

This is the expression which we have already obtained for the area of a triangle in terms of its sides. Art. 6. (3).

11. *To find an expression for the area of a regular polygon*—

(1) In terms of the number of sides and the radius of the circumscribed circle.

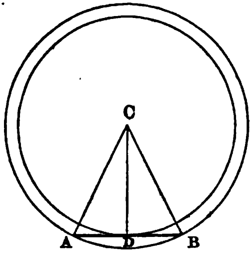
(2) In terms of the number of sides and the radius of the inscribed circle.

(3) In terms of the number of sides and the length of one side.

By a *regular* polygon we mean a polygon which has all its sides equal to each other, and also all its angles equal.

It will be manifest that if any two angles of a regular polygon be bisected by straight lines, these lines will intersect in a point which will be equidistant from all the angular points of the polygon, and therefore also equidistant from the middle points of all the sides.

Hence, with this point as centre, two circles can be described, one of which passes through all the angular points of the polygon, and is called the circumscribed circle; and the other touches all the sides in their middle points, and is called the inscribed circle.



In the figure, AB represents one side of the polygon, D its middle point, and C the common centre of the two circles; then $CA=CB$, and CD is perpendicular to AB; also the angle ACB, at the centre, is the same for each side of the polygon.

Let n be the number of sides of the polygon,

a the length of one side $= AB$,

R the radius of the circumscribed circle $= CA$,

r........................ inscribed circle $= CD$,

and A the area of the polygon.

Then the angle $ACB = \frac{2\pi}{n}$, or $\frac{360^\circ}{n}$, according as it is measured by circular measure or by degrees.

Case (1). *Area in terms of the number of sides and the radius of the circumscribed circle.*

From the figure, we have

$$\text{area of polygon} = n \text{ times area of triangle } ACB$$

$$= n\,\frac{AC \times BC}{2} \sin ACB, \text{ Art. 6. (2)},$$

$$\therefore\ A = n\,\frac{R^2}{2} \sin\frac{2\pi}{n}, \quad \text{or } A = n\,\frac{R^2}{2} \sin\frac{360^\circ}{n};$$

$$\therefore\ \log A = \log n + 2\log R + \log\sin\frac{360^\circ}{n} - \log 2.$$

Case (2). *Area in terms of the number of sides and the radius of the inscribed circle.*

Referring to the figure, we have, as before,

$$\text{area of polygon} = n \text{ times area of triangle } ACB$$

$$= n\,\frac{AB \times CD}{2} \qquad \text{Art. 6. (1)},$$

$$= n\,AD \times CD,$$

$$\text{but } AD = CD \,.\, \tan ACD = r\tan\frac{\pi}{n}, \quad \text{or } r\tan\frac{180^\circ}{n};$$

$$\therefore\ A = nr^2\tan\frac{\pi}{n}, \text{ or } A = nr^2\tan\frac{180^\circ}{n};$$

$$\text{and } \log A = \log n + 2\log r + \log\tan\frac{180^\circ}{n}.$$

Case (3). *Area in terms of the number of sides and the length of one side.*

As in the preceding case,

$$\text{area of polygon} = nAD \times CD.$$

$$\text{Now } AD = \frac{a}{2},\quad CD = AD\cot ACD,$$

$$\therefore\ CD = \frac{a}{2}\cot\frac{\pi}{n},\ \text{ or } \frac{a}{2}\cot\frac{180^\circ}{n};$$

$$A = n\frac{a^2}{4}\cot\frac{\pi}{n},\ \text{ or } A = n\frac{a^2}{4}\cot\frac{180^\circ}{n};$$

$$\text{and } \log A = \log n + 2\log a + \log\cot\frac{180^\circ}{n} - \log 4.$$

Cor. The perimeter of the polygon will evidently be $n\,.\,AB$.

$$\therefore\ \text{perimeter} = na,\quad \text{or } 2nR\sin\frac{\pi}{n},\quad \text{or } 2nr\tan\frac{\pi}{n}.$$

12. *To find the area of the surface of a right prism.*

Def. A prism is a solid figure contained by plane figures, of which two which are opposite are equal, similar and parallel to each other; and the rest are parallelograms. Euc. xi. Def. 13.

When these parallelograms are rectangles, the solid is said to be a *right* prism.

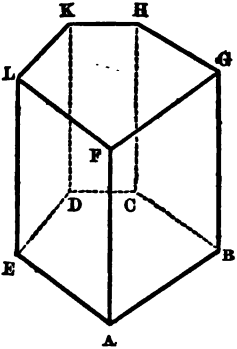
Let $ABCDE\,.\,FGHKL$ represent a right prism.

The area of the lateral surface will be the sum of the areas of the rectangles AG, BH, CK, DL, EF.

$$\therefore \text{ area} = AB \times BG + BC \times CH + CD \times DK + DE \times EL + EA \times AF$$

$$= (AB + BC + CD + DE + EA)\, AF,$$

$\because$ BG, CH, DK, EL are each equal to AF;



but $AB + BC + CD + DE + EA$ = perimeter of base,

and AF = altitude of prism;

$\therefore$ area of lateral surface = perimeter of base × altitude.

If we wish to find the area of the whole surface we must add to this the areas of the two ends, or, since they are equal, twice the area of one end.

Let A = area of one end,
p = perimeter of base,
h = altitude, or length of the prism,
S = area of the whole surface;

$$\therefore S = 2A + ph.$$

This result will manifestly be true whatever be the number of sides of the prism.

13. *To find the area of the surface of an oblique prism.*

If in the figure of the preceding article we suppose the edges of the prism to be *inclined* to the base, the solid will be an *oblique* prism, and the figures *AG*, *BH*, &c. will be parallelograms having a common length equal to *AF*.

If now we suppose a plane perpendicular to *AF* to intersect the prism, it will be manifest that the area of the lateral surface of the prism will be equal to the perimeter of this section, multiplied by the length of *AF*.

Hence the formula

$$S = 2A + ph,$$

when p represents the perimeter of the *transverse section*, will give the area of the whole surface of an *oblique* prism.

14. *To find the area of the surface of a Frustum of a Prism.*

If a Prism be cut by two planes inclined to each other at any angle, the solid included between these planes is called a *Frustum* of the Prism.

The figure of Art. 12 will represent a Frustum of a Prism, if we suppose the planes *ABE*, *FGL* to be *inclined* to each other.

All the figures *AG*, *BH*, &c. will then be trapezoids, and if h_1, h_2, h_3, h_4, h_5 be the lengths of the edges taken in the order *ABCDE*, beginning with *AF*, and p_1, p_2, p_3, p_4, p_5 be the sides of the perimeter of the *transverse section* in the same order, beginning from *AF*,

$$\therefore \text{ 2 area of lateral surface} = p_1(h_1 + h_2) + p_2(h_2 + h_3)$$
$$+ p_3(h_3 + h_4) + p_4(h_4 + h_5) + p_5(h_5 + h_1).$$

Hence this rule: "Multiply each side of the perimeter of the transverse section by the sum of the two edges in which it terminates, and the sum of the products will be twice the area."

This rule will manifestly hold for a Frustum of a Prism of any number of sides.

Cor. If the transverse section be a *regular* polygon, of n sides suppose, the prism will be a *regular* prism, and if p be the perimeter of the section, each of the quantities

$p_1, p_2,$ &c. will be equal to $\frac{p}{n}$;

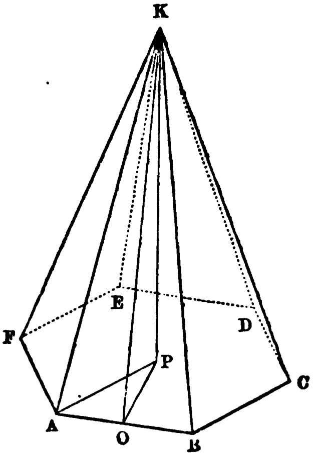
$$\therefore \text{ area of lateral surface} = p\,\frac{\text{sum of the edges}}{n}.$$

The quantity $\frac{\text{sum of the edges}}{n}$ is called the *mean* length of the Frustum: it is the straight line, parallel to the edges, which passes through the centre of the transverse section, and is terminated by the two inclined ends of the solid. It is frequently called the *axis* of the solid.

15. *To find the area of the surface of a regular pyramid.*

Def. 1. A pyramid is a solid figure contained by a number of plane triangles, all of which have a common vertex, and their bases form a plane polygon.

Def. 2. A *regular* pyramid is contained by a number of equal *isosceles* triangles, and their bases form a *regular* polygon.



Let $ABCDEF$ be the base, and K the vertex of a regular pyramid.

Then all the triangles AKB, BKC, CKD, &c. are equal to each other, and the area of the pyramid will be equal to the sum of the areas of all these triangles together with the area of the base.

Bisect AB in O and join KO; then

$$\text{area of the triangle } AKB = \frac{AB \times OK}{2}.$$

Therefore area of the inclined surface of the pyramid

$$= \frac{OK}{2}(AB + BC + CD + DE + EF + FG)$$

$$= \frac{OK}{2} \times \text{perimeter of base.}$$

Let A = area of the base,

p = perimeter of the base,

$h = OK$,

S = area of the whole surface;

$$\therefore S = A + \frac{ph}{2}.$$

This result will manifestly be true whatever be the number of sides of the pyramid.

The line OK is called the slant height of the pyramid, and KP, which is perpendicular to the base, the altitude.

16. *To find the area of the surface of a frustum of a regular pyramid, made by a plane parallel to the base.*

In this solid the base and top will be similar regular polygons, and the inclined faces similar and equal trapezoids.

Let h be the slant height of the frustrum, n the number of sides, and P, p the perimeters of the base and top;

$$\therefore\ 2 \text{ area of each face} = h\left(\frac{P}{n} + \frac{p}{n}\right);$$

$$\therefore\ 2 \text{ area of inclined surface} = h\,(P + p).$$

Cor. If A_1, A_2 be the areas of the base and top,

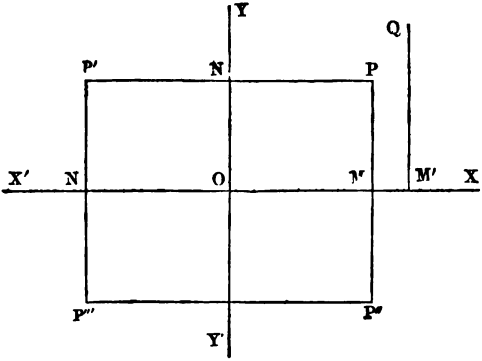
$$\therefore\ \text{area of whole surface} = A_1 + A_2 + \frac{h}{2}\,(P + p).$$

When the pyramid is *irregular*, we can only find the area of the surface by taking the sum of the areas of the triangles which contain the pyramid; and a similar remark applies to the area of the surface of a frustum of an irregular pyramid.

CHAPTER II.

METHOD OF COORDINATES, AND ITS APPLICATION TO THE DETERMINATION OF THE AREAS OF PLANE FIGURES.

17. If we take any point O, and through it draw any two straight lines XOX', YOY', at right angles to each other, we can determine the position of any point P in the plane of OX and OY, by measuring the lines PM, PN, drawn from P, perpendicular to OX and OY respectively; or, since $PN = OM$, by measuring the lines OM and MP.



The former of these lines is called the *abscissa*, and the latter the *ordinate* of the point P, and the two together are called the *coordinates* of the point.

The *abscissæ* are represented, generally, by the letter x, and the *ordinates* by the letter y, and we distinguish one point P from another point Q by saying that the coordinates of P are $x_1\ y_1$, and those of Q, $x_2\ y_2$, which simply means that $OM=x_1$, $MP=y_1$, $OM'=x_2$, and $M'Q=y_2$.

The two fixed lines XX' and YY' are called the *axes of coordinates*, and the point O, in which they intersect, is called the *origin*.

It will be observed that there are *four* points which will have the same coordinates as the point P, so far as we have already defined coordinates, and that these points may be determined in the following manner.

Produce PN to P' making $NP'=NP$,

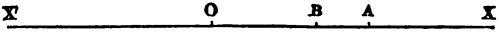
produce PM to P'' .... $MP''=MP$,

and complete the rectangle PP'''.

These four points P, P', P'' and P''' will evidently have coordinates of the same magnitude as those of the point P.

It therefore becomes necessary to distinguish between the coordinates of these four points, and to do this we must take into consideration the *directions* in which the coordinates are drawn as well as their magnitudes.

We shall first show that if a straight line measured in any direction be considered positive, a straight line measured in the opposite direction *must* be considered negative.



Take the straight line XX'; and suppose a line OA measured from O to the right hand, to be denoted by a, and let OB, also measured from O, be denoted by x, and $=b$.

Then since $OB = OA - BA$,

$$\therefore\ x = a - b.$$

Now so long as b is less than a, x will be positive, and the point B will be on the *right* of O.

When $b = a$, $x = 0$,

and the point B will coincide with O.

When $b > a$, x will be negative,

and the point B will be on the *left* of O.

Hence we see that if a line, represented by an algebraical symbol, be affected with a *negative* sign, it must be measured in a direction opposite to that in which it would have been measured if it had been affected with a *positive* sign.

Now, suppose that lines measured from YY' to the *right-hand* are affected with a positive sign; those measured from the same line to the *left-hand* will be affected with a negative sign.

Similarly, lines measured from XX' *upwards* being considered positive, those measured from XX' *downwards* will be negative.

It is arbitrary in which direction the positive lines are measured, but it is usual to take them as we have stated.

Now let a and b denote the *numerical values* of the lines OM and MP.

Then the coordinates of P will be

$$x = +a,$$
$$y = +b,$$

those of P'

$$x = -a,$$
$$y = +b,$$

those of P''

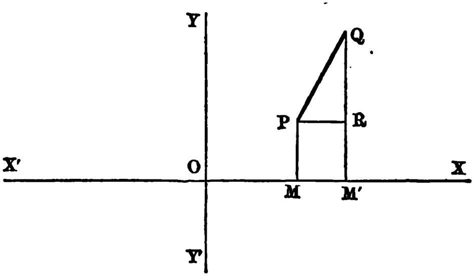
$$x = +a,$$
$$y = -b,$$

and those of P'''

$$x=-a,$$
$$y=-b,$$

and since no two of these pairs of coordinates are the same, we are thus enabled to distinguish between the four points P, P', P'' and P'''.

18. *To find the distance between two points in terms of their coordinates.*



Let P and Q be the two points, the distance between which is to be found.

Take any axes of coordinates OX, OY, at right angles,

draw PM, QM' parallel to OY,

and PR ....OX.

Then PQR is a right-angled triangle,

$$\therefore PQ^2=PR^2+QR^2. \quad \textit{Euclid, } \text{I. } 47.$$

Let $OM=x_1$, $MP=y_1$, be coordinates of P,

$OM'=x_2$, $M'Q=y_2$, ... Q;

$$\therefore PR=MM'=OM'-OM=x_2-x_1,$$

$$\text{and } QR=QM'-RM'=QM'-PM=y_2-y_1,$$

hence, if we denote PQ by δ, we have by substitution in the above equation,

$$\delta^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

$$\therefore\ \delta = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

in which we take the positive sign, because we are here only concerned with the *absolute* distance between P and Q.

The student should observe that although we have taken for a model the case in which x_1, y_1, x_2 and y_2 are all *positive*, and x_2 and y_2 greater than x_1 and y_1 respectively, yet, by proper attention to the *signs* of the coordinates the formula for δ will include every case.

Take, for example, the distance between P and P''' in the figure of Art. 17.

Here

$$\begin{aligned} PP'''^2 &= PP'^2 + P'P'''^2 \\ &= (2x)^2 + (2y)^2 \\ &= 4(x^2 + y^2), \end{aligned}$$

and, by the formula,

$$\begin{aligned} \delta^2 = PP'''^2 &= \{x - (-x)\}^2 + \{y - (-y)\}^2 \\ &= (x + x)^2 + (y + y)^2 \\ &= (2x)^2 + (2y)^2 \\ &= 4(x^2 + y^2), \text{ as before.} \end{aligned}$$

19. *To find an expression for the area of a triangle in terms of the coordinates of its angular points.*

Let ABC be any triangle, OX, OY axes of coordinates.

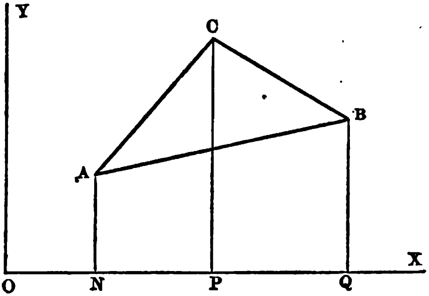
Let x_1, y_1, be the coordinates of A,

x_2, y_2, ... B,

x_3, y_3, ... C,

and let S be the area of the triangle.

Now area of $ABC =$ area of $ANPC +$ area of $CPQB$ $-$ area of $ANQB$.



$$\therefore\ S = \tfrac{1}{2} NP(AN + CP) + \tfrac{1}{2} PQ(CP + BQ) - \tfrac{1}{2} NQ(AN + BQ);$$

$$\therefore\ 2S = (x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)$$

$$= x_3 y_1 - x_1 y_1 + x_3 y_3 - x_1 y_3 + x_2 y_2 + x_2 y_3 - x_3 y_2 - x_3 y_3 - x_2 y_1 - x_2 y_2 + x_1 y_1 + x_1 y_2$$

$$= x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3.$$

This expression is very important, and, by considering its form, may be easily remembered.

We observe that each term is of the form xy, and that there is a pair of these terms, with the sign $-$ between them, for each point of the triangle. Hence, if we write down the expression $xy - xy$ three times, thus,

$$xy - xy + xy - xy + xy - xy,$$

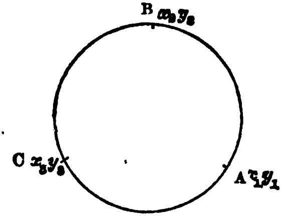
and add the suffices in the order indicated, viz.

1, 2 2, 1 ; 2, 3 3, 2 ; 3, 1 1, 3,

we shall have the expression for $2S$, viz.,

$$2S = x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3.$$

The reader must be careful always to take the coordinates in the *same order of succession.* Thus, we pass from x_1 to x_2, from x_2 to x_3, and from x_3 to x_1, as indicated in the figure,



and *not* from x_1 to x_2, x_3 to x_2, x_3 to x_1, where the order of succession would be violated in passing from x_3 to x_2.

Cor. 1. If we make the origin of the coordinates to coincide with one point of the triangle, the point A suppose, then $x_1=0$, $y_1=0$, and the expression for the area becomes

$$2S=x_2 y_3-x_3 y_2.$$

Cor. 2. If we also make one of the coordinate axes to coincide with one side of the triangle, the axis OX coinciding with AB suppose,

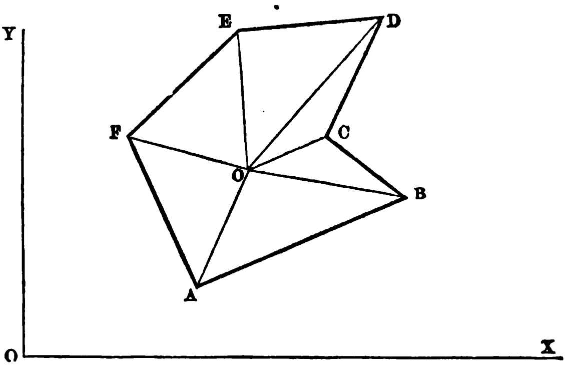
then also $y_2=0$, and we have

$$2S=x_2 y_3.$$

It will be seen that this is the same thing as the base multiplied by the perpendicular.

20. *To find an expression for the area of any rectilinear figure in terms of the coordinates of its angular points.*

Let $ABCDEF$ be any rectilinear figure; OX, OY, axes of coordinates.



Let	$x_1\,y_1$ be the coordinates of	A,
	$x_2\,y_2$	B,
	$x_3\,y_3$	C,
	$x_4\,y_4$	D
	$x_5\,y_5$	E,
	$x_6\,y_6$	F,
and	$x\,y$	O,

any point *within* the figure.

Join OA, OB, OC, OD, OE and OF, dividing the figure into as many triangles as it has sides. Then the area of the whole figure will be equal to the sum of the areas of all these triangles.

Now, by the preceding Article,

$$2\text{ area }AOB = x_1\,y_2 - x_2\,y_1 + x_2\,y - x\,y_2 + x\,y_1 - x_1\,y,$$
$$2\text{ area }BOC = x_2\,y_3 - x_3\,y_2 + x_3\,y - x\,y_3 + x\,y_2 - x_2\,y,$$
$$2\text{ area }COD = x_3\,y_4 - x_4\,y_3 + x_4\,y - x\,y_4 + x\,y_3 - x_3\,y,$$
$$2\text{ area }DOE = x_4\,y_5 - x_5\,y_4 + x_5\,y - x\,y_5 + x\,y_4 - x_4\,y,$$
$$2\text{ area }EOF = x_5\,y_6 - x_6\,y_5 + x_6\,y - x\,y_6 + x\,y_5 - x_5\,y,$$
$$2\text{ area }FOA = x_6\,y_1 - x_1\,y_6 + x_1\,y - x\,y_1 + x\,y_6 - x_6\,y.$$

Adding up, we observe that the third, fourth, fifth and sixth columns on the right-hand side will disappear from the result, and we shall have for twice the area of the whole figure *ABCDEF*, the sum of the first and second columns only; or, denoting by *S* the area of *ABCDEF*,

$$2S = x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3$$
$$+ x_4 y_5 - x_5 y_4 + x_5 y_6 - x_6 y_5 + x_6 y_1 - x_1 y_6.$$

This expression is of the same form as that already obtained for a triangle, and can be as easily remembered.

We may here remark that it is independent of the coordinates of the point *O*, as it ought to be, for the area must be the same in whatever manner we may divide the figure into triangles. Whatever may be the number of angular points in the rectilinear figure, there will be the *same number* of terms *of the form* $(x_n y_{n+1} - x_{n+1} y_n)$ in the expression for twice the area, which may therefore be obtained by giving to n the values, 1, 2, 3, &c. in succession, and taking the sum of the resulting quantities.

We may take any point of the figure to start from, but the remaining points must be taken in their order of succession, as explained in the preceding Art. This is most important, for, if not attended to, the result will *not* be the expression for twice the area.

As in the case of the triangle we may take any point of the figure for the origin of the coordinates, and we may also make one of the axes of coordinates to coincide with one side of the figure.

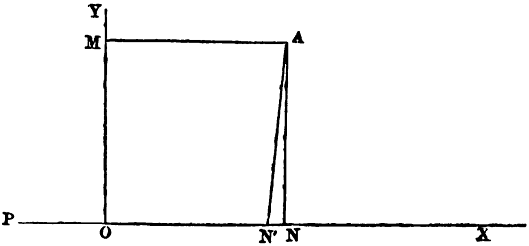
This will reduce the expression for the area to fewer terms, but will at the same time destroy its symmetry. Thus, suppose we take *A* for the origin of coordinates, and *AB* for the axis of x, then will $x_1 = 0$, $y_1 = 0$ and $x_2 = 0$, and therefore all those terms which multiply x_1, x_2, y_1, will disappear from the expression for the area.

The origin of coordinates may be taken anywhere, and the coordinate axes drawn through it in any two directions at right angles which may be most convenient. If either or both of them be drawn across the figure it is evident that one or more of the coordinates will be negative. This will not affect the expression for the area if attention be paid to the algebraical signs of the coordinates.

It will frequently happen that the expression for an area will come out negative. In all such cases we discard the sign, and the reason for doing so can be easily explained.

The expression for an area is, in all cases, the difference between a set of positive and a set of negative terms. Now if in any case we find the expression for the area negative when we take the angular points in the order *A*, *B*, *C*, *D*, *E*, *F*, (see figure) we shall necessarily find it positive when we take the points in the order *A*, *F*, *E*, *D*, *C*, *B*; now, since the order in which we take the points is arbitrary, we have a sufficient reason for discarding the negative sign when it happens to appear.

21. In applying the method of coordinates to find the area of any enclosure it is not necessary that any point of the enclosure should be accessible. All that is absolutely necessary is that we should have access to two straight lines at right angles to each other.



Thus, to determine the coordinates x_1, y_1, of any point *A*, we go along the line *OX* and fix the point *N*, at which *AN* and *ON* are at right angles to each other, by means of any convenient instrument, and we then measure the distance *ON* which gives x_1. In the same manner we determine the point *M* and measure *OM* which gives y_1, and thus the coordinates of every point may be found without leaving the coordinate axes. A very convenient instrument for this purpose is the *pocket sextant;* for, if the index be set at 90°, the angle ONA will be a right angle whenever the objects which mark the points O and A are seen to coincide, and it then only remains to keep the point N accurately in the line OX. This is easily done by previously setting up a pole at a point P, in the line XO *produced*, and at some distance from the pole which marks the point O. When these *two* poles and the object at A appear to coincide in the instrument, the point N is determined with considerable accuracy. The *optical square* is an instrument which may also be used for the same purpose.

22. Although the method just described of obtaining the coordinates of the angular points of an enclosed area is theoretically perfect, and must be adopted whenever the points are inaccessible; yet, whenever we can obtain access to these points, it will be preferable to measure the coordinates from the points themselves: thus, in the figure, we should obtain y_1 by measuring the line AN rather than by measuring OM, because there will be less liability to error in the former case. This we shall now show.

Suppose in attempting to determine the point N, (at which ON, AN are accurately at right angles) we take the point N' instead; we make an error NN' in x_1. The corresponding error in y_1 will be $AN'-AN$, or $y-y_1$ if we put y for AN'. Now $y^2=y_1^2+NN'^2$ because ANN' is a right angle;

$$\therefore\ y^2-y_1^2 \text{ or } (y-y_1)(y+y_1)=NN'^2;$$

$$\therefore\ y-y_1=\frac{NN'}{y+y_1}.NN'.$$

Now NN' will always be very small compared with $y+y_1$, and therefore the error in y_1 will be *very much smaller* than the error in x_1. Thus with the same degree of care in determining the points N, M, we shall obtain more accurate values of the coordinates by measuring them from the point A, than from the point O.

We shall now show how the methods which we have investigated may be applied in practice.

23. Given the following values in feet of the coordinates of the angular points of a triangle, viz.,

$$x_1=83, \qquad x_2=134, \qquad x_3=-61\cdot5,$$
$$y_1=21\cdot7, \qquad y_2=115, \qquad y_3=54;$$

find the area of the triangle, and the length of any one side; also, construct the triangle in position with respect to the coordinate axes.

We have, from the general formula,

$$\begin{aligned} 2\text{ area} &= x_1y_2-x_2y_1 \\ &\quad +x_2y_3-x_3y_2 \\ &\quad +x_3y_1-x_1y_3; \end{aligned}$$

$$\begin{aligned} \therefore\ 2\text{ area} &= 83\times115-134\times21\cdot7 \\ &\quad +134\times54-(-61\cdot5)\times115 \\ &\quad +(-61\cdot5)\times21\cdot7-83\times54; \end{aligned}$$

$$\begin{aligned} \therefore\ 2\text{ area} &= 23853\cdot5-8724\cdot35 \\ &= 14939\cdot15; \end{aligned}$$

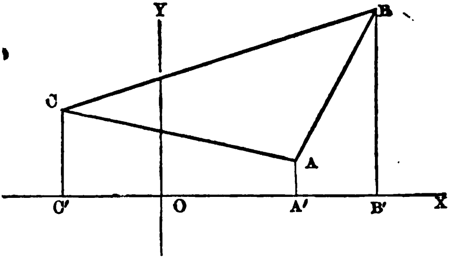
$$\begin{aligned} \therefore\ \text{area} &= 7564\cdot575 \text{ square feet} \\ &= 840 \text{ sq. yds., } 4 \text{ sq. ft., } 82\cdot8 \text{ sq. ins.} \end{aligned}$$

To find the length of any one side, say the side terminating in the points $(x_1\ y_1)$ and $(x_2\ y_2)$, we have, if c represent the length,

$$\begin{aligned} c &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(83-134)^2+(21\cdot7-115)^2} \\ &= \sqrt{(-51)^2+(-93\cdot3)^2} \\ &= \sqrt{13305\cdot89}=115\cdot75 \text{ very nearly,} \end{aligned}$$

and, similarly, the length of any other side may be found.

To construct the triangle, draw two straight lines *OX*, *OY*, at right angles.



In the direction *OX*, set off $OA'=83$, $OB'=134$, and in the opposite direction, $OC'=61 \cdot 5$, and draw *A'A*, *B'B*, *C'C* parallel to *OY*, and equal, respectively, to 21·7, 115, and 54. These three points *A*, *B*, *C*, being joined, the construction will be completed.

24. The following table represents a page of a note-book, showing how the coordinates of the angular points of an enclosed piece of ground may be entered and registered. The first and third columns contain the suffices of *x* and *y* in their order of succession, and the second and fourth the values of the corresponding coordinates.

Thus, $x_4=-6 \cdot 1$, $x_8=12 \cdot 4$, $y_1=0$.

The unit of length is here supposed to be a chain, of 66 feet, or 100 links.

x		*y*	
1	0	1	0
2	7·3	2	10
3	4·5	3	16·3
4	− 6·1	4	11·2
5	−13	5	9·31
6	−17	6	0
7	− 8	7	− 7·4
8	12·4	8	−12
9	18	9	− 7
10	11·7	10	2·3

We shall take these data and calculate the area of the enclosure, and shall then apply them to construct a plan of the same.

Now,
$$\begin{aligned} 2\,\text{area} = {} & x_1 y_2 - x_2 y_1 \\ & + x_2 y_3 - x_3 y_2 \\ & + \ldots\ldots\ldots\ldots\ldots \\ & + x_{10} y_1 - x_1 y_{10} \end{aligned}$$

$+\ldots\ldots$, taking the place of the intermediate terms;

$$\begin{aligned} \therefore\ 2\,\text{area} = {} & 0 \times 10 - 7{\cdot}3 \times 0 \\ & + 7{\cdot}3 \times 16{\cdot}3 - 4{\cdot}5 \times 10 \\ & + 4{\cdot}5 \times 11{\cdot}2 - (-6{\cdot}1) \times 16{\cdot}3 \\ & + (-6{\cdot}1) \times 9{\cdot}31 - (-13) \times 11{\cdot}2 \\ & + (-13) \times 0 - (-17) \times 9{\cdot}31 \\ & + (-17) \times (-7{\cdot}4) - (-8) \times 0 \\ & + (-8) \times (-12) - 12{\cdot}4 \times (-7{\cdot}4) \\ & + 12{\cdot}4 \times (-7) - 18 \times (-12) \\ & + 18 \times 2{\cdot}3 - 11{\cdot}7 \times (-7) \\ & + 11{\cdot}7 \times 0 - 0 \times 2{\cdot}3. \end{aligned}$$

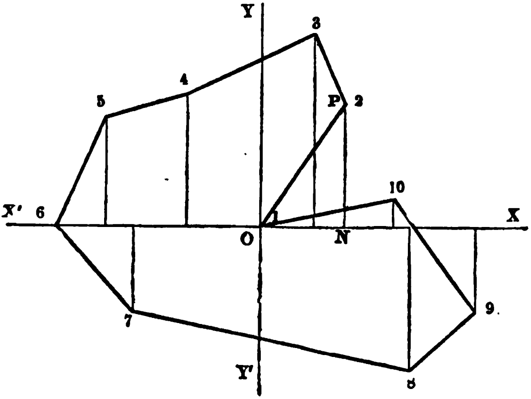
Multiplying these numbers together, and adding up, we find

$$\begin{aligned} 2\,\text{area} & = 1225{\cdot}45 - 187{\cdot}871 \\ & = 1037{\cdot}579\,; \end{aligned}$$

$$\begin{aligned} \therefore\ \text{area} & = 518{\cdot}7895 \text{ sq. chains}, \\ & = 51{\cdot}87895 \text{ acres} \\ & = 51 \text{ a.}, 3 \text{ r.}, 20 \text{p.}, 7 \text{ yds.} \end{aligned}$$

The reader should observe that in the two columns of products in the expression for 2 area, the values of x_1, x_2 ...., y_1, y_2 ...., will occur in each vertical column in the same order in which they are entered in the note-book. By attention to this, errors may be readily detected before the multiplications are performed.

We shall now construct the figure the area of which we have just found.



Taking XOX' and YOY' at right angles, we see that the first point will be at O, because $x_1=0, y_1=0$. For the second point P, marked 2, we measure $ON=73$, and $NP=10$; and in the same manner we lay down the points 3, 4, 5, &c.; the point 6 being on the line XX' because $y_6=0$. When all the points have been thus laid down, we join them by straight lines in their order of succession, and then the construction will be completed.

We may also find the length of any side of the figure, or, the distance between any two of the angular points. Thus, we may find the distance between the points marked 5, and 9.

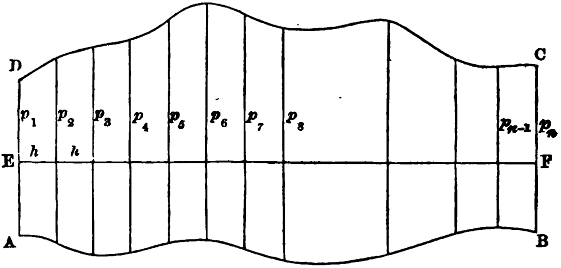
$$\begin{aligned}\text{This distance} &= \sqrt{(x_5-x_9)^2+(y_5-y_9)^2}\\ &= \sqrt{(-13-18)^2+(9{\cdot}31-(-7))^2}\\ &= \sqrt{1227{\cdot}0161}=35{\cdot}025\\ &= 35 \text{ chains, 3 links, very nearly.}\end{aligned}$$

Such a line as this may also be measured, where practicable, and compared with its calculated length, and will thus become a test of the accuracy with which the coordinates have been measured.

CHAPTER III.

APPROXIMATION TO THE AREAS OF PLANE CURVILINEAR FIGURES.

25. *To find approximately the area of a plane figure bounded by any curve.*



Let *ABCD* be a portion of a plane figure bounded by the curved lines *AB*, *CD*, and the two *parallel* lines *AD*, *BC*.

Draw *EF* perpendicular to *AD* or *BC* through any convenient point in *AD* or *BC*.

Divide *EF* into any number of equal parts, say n, and let each of these parts be represented by h, so that

$$EF = nh.$$

Through the points of division draw lines parallel to *AD*, or perpendicular to *EF*, terminated by the boundaries *AB*, *CD*; and let these parallels, taken in order, be represented by $p_1, p_2, \ldots\ldots p_{n-1}, p_n$, *AD* being the first and *BC* the last, as shown in the figure.

Now the area contained between any two consecutive parallels or ordinates, and the portions of the curved lines between them, will be very nearly a trapezoid; and the more nearly, the closer the parallels are taken to each other.

Hence, approximately,

$$\text{area between } p_1 \text{ and } p_2 = h\,\frac{p_1+p_2}{2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ p_2 \ldots\ldots p_3 = h\,\frac{p_2+p_3}{2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ p_{n-1} \ldots\ldots p_n = h\,\frac{p_{n-1}+p_n}{2};$$

$$\therefore \text{ area of } ABCD = h\left(\frac{p_1+2p_2+2p_3+\ldots 2p_{n-1}+p_n}{2}\right)$$

$$= h\left(\frac{p_1+p_n}{2}+p_2+p_3+\ldots p_{n-1}\right).$$

Hence we have this simple rule,

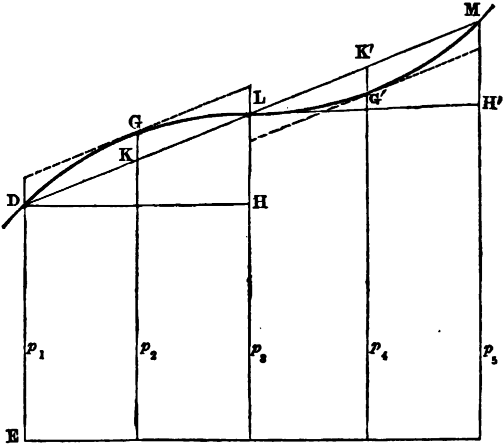
"Add half the sum of the extreme ordinates to the sum of all the intermediate ordinates, and multiply the result by the distance between two consecutive ordinates; the product will be the area, approximately." In applying this method in practice, the line *EF* should be measured and divided by *n* to obtain *h*; for, any error in the measurement will then be divided by *n*, and the result will be more accurate than if *h* were measured directly.

26. We shall now investigate a method, generally known as "Simson's Rule," by which a much closer degree of approximation to the area of a curve may be obtained.

We shall make use of the figure of the preceding article, but it will also be convenient to exhibit a portion of it on a larger scale.

Let the following figure represent that portion which is contained between the straight line *EF*, the parallels p_1

and p_5, and the portion of the curve intercepted between them.



Now, area between p_1 and p_3 = trapesoid *EL* + curvilinear area *DGL*. But since the curvature of *DGL* is supposed to be small, and since the limiting form of every curve of finite curvature is a parabola (Newton, *Lemma* XI. Cor. 5), we shall obtain a close degree of approximation to the curvilinear area *DGL* by supposing the curve *DGL* a portion of a parabola, of which *GK* is a diameter, and the dotted line through *G* parallel to *DL* a tangent at its extremity.

It will be shown in a subsequent chapter that the parabolic area *DGL* is two-thirds of the parallelogram *DGL*, *i.e.* $\frac{2}{3}GK \times DH$.

Hence, area between p_1 and p_3

$$=h(p_1+p_3)+\tfrac{2}{3}\{p_2-\tfrac{1}{2}(p_1+p_3)\}2h$$

$$=\frac{h}{3}(p_1+p_3+4p_2).$$

For the area between p_3 and p_5 we shall have to *subtract* the curvilinear area *LG'M* from the trapezoid;

∴ area between p_3 and p_5

$$=h(p_3+p_5)-\tfrac{2}{3}\{\tfrac{1}{2}(p_3+p_5)-p_4\}\,2h$$

$$=\frac{h}{3}(p_3+p_5+4p_4),$$

which shows that the same formula is applicable whether the curve be *concave* or *convex* to the straight line *EF*.

Hence if we suppose the whole line *EF* divided into an *even* number of equal parts, say $2n$, the last parallel will be represented by p_{2n+1}, and we shall have, very approximately,

$$\text{area between } p_1 \text{ and } p_3=\frac{h}{3}(p_1+p_3+4p_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots p_3\ldots\ldots p_5=\frac{h}{3}(p_3+p_5+4p_4),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots p_{2n-1}\ldots\ldots p_{2n+1}=\frac{h}{3}(p_{2n-1}+p_{2n+1}+4p_{2n}),$$

$$\therefore \text{ whole area}=\frac{h}{3}\{p_1+p_{2n+1}+2(p_3+p_5+\ldots+p_{2n-1})$$
$$+4(p_2+p_4+\ldots+p_{2n})\}.$$

Hence this rule.

"Add together the extreme ordinates, twice the sum of all the intermediate ordinates of an *odd* order, and four times the sum of all the intermediate ordinates of an *even* order, and multiply the result by one-third of the distance between two consecutive ordinates."

27. We are now able to find the area of a plane figure bounded by any curve, to a close degree of approximation.

We first inscribe within the figure a rectilinear polygon as nearly as possible coinciding with it, and find the area of this polygon by the method of coordinates or otherwise.

We then find the areas of the remaining curvilinear portions by "Simson's Rule," and add or subtract these areas to or from that of the polygon according as they happen to be without or within it.

28. The assumption made in Art. 25 is equivalent to supposing DG, GL, &c. (see the fig. of Art. 26) to be *straight* lines instead of *curves*.

Hence, if x, y be the coordinates of any point in the straight line DG,

$$y = p_1 + x \tan GDH,$$

i. e. y may he expressed by the formula

$$y = A + Bx,$$

where A and B are constant.

Again, in Art. 26, the assumption that the curve DGL is a parabolic arc is equivalent to supposing that any ordinate between p_1 and p_3 can be expressed by the formula

$$y = A + Bx + Cx^2,$$

where A, B, C are constants, since it is a property of every parabolic arc that its ordinate can be expressed in this form when the axis of the parabola is perpendicular to EF.

In Art. 25 we suppose the two parts into which each of the ordinates is divided by the straight line EF to be known, otherwise the figure could not be constructed, although its area might be calculated.

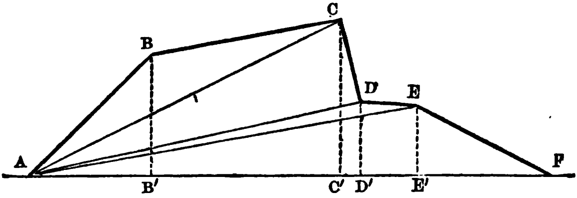
It is desirable to avoid great differences between consecutive ordinates: where the curve is very irregular the ordinates must be taken closer together.

We shall recur to this subject in a subsequent chapter, where we shall give another proof of "Simson's Rule," and shall also investigate two other remarkable formulæ by which the area of any curvilinear figure may be obtained to any required degree of approximation.

CHAPTER IV.

ON THE AREAS OF THE SECTIONS OF CERTAIN MILITARY EARTHWORKS.

29. *To find the area of the transverse section of a parapet.*



Let $ABCDEF$ represent a transverse section of a parapet.

Draw BB', CC', DD', EE' perpendicular to the ground-line AF, dividing the figure into triangles and trapezoids, the areas of which can be readily found from the dimensions of the parapet, and their sum will be the area of the section.

Again, by drawing the straight lines AC, AD, AE, we divide the figure into triangles all of which have a common vertex at the point A, and the areas of these triangles may be found by the method of coordinates, as follows.

Let $AB'=x_1$, $AC=x_2$, $AD'=x_3$, $AE'=x_4$, $AF=x_5$,
$BB'=y_1$, $CC'=y_2$, $DD'=y_3$, $EE'=y_4$, $y_5=0$.

Then 2 area $ABC = x_1 y_2 - x_2 y_1$,

...... $ACD = x_2 y_3 - x_3 y_2$,

...... $ADE = x_3 y_4 - x_4 y_3$,

...... $AEF = \qquad - x_5 y_4$, since $y_5 = 0$;

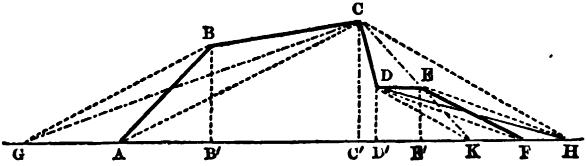
$$\therefore \text{ 2 area of section} = x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3 - x_5 y_4.$$

This is, of course, the same expression which we should obtain by treating the section as a polygon, without dividing it into triangles.

The particular division into triangles which we have just employed will be very useful hereafter when we have to find the *volume* of any portion of a parapet.

It should be observed that the coordinates $x_1\ y_1$ &c. are always known, and are, in fact, the dimensions from which a parapet is constructed.

30. *The area of the section of a parapet may also be found by reducing it to a triangle, as follows:*



Let *ABCDEF* be a section of a parapet as before.

Join *CA*, draw *BG* parallel to it and join *CG*.

Then the triangles *CBA*, *CGA* being on the same base *CA* and between the same parallels *CA*, *GB*, are equal in area; therefore the portion *CBAC′* of the section is equal in area to the triangle *CGC′*.

Similarly, if we join *DF*, draw *EH* parallel to it and join *DH*, the triangle *DEF* will be equal to the triangle *DHF*; and lastly, if we join *CH*, draw *DK* parallel to it and join CK, the triangle DHK will be equal to the triangle DCK, and consequently, the portion $CDEFC'$ of the section will be equal to the triangle CKC'.

Hence the area of the whole section equals the area of the triangle GCK.

If the figure be drawn to scale GK may be measured, and the area of the section will be $\frac{1}{2}GK \times CC'$.

It is not, however, necessary that the figure should be drawn to scale, for the length of GK can be easily obtained by calculation, as follows.

By similar right-angled triangles

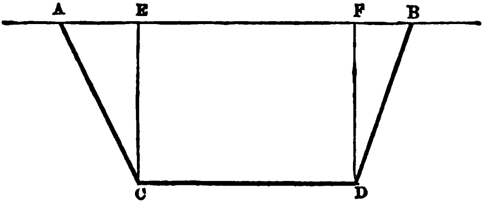
$$\frac{GB'}{BB'} = \frac{AC'}{CC'}, \quad \frac{HE'}{EE'} = \frac{FD'}{DD'}, \quad \frac{KD'}{DD'} = \frac{HC'}{CC'};$$

from these we obtain, successively,

GB', HE' and KD', all the other dimensions being given,

and knowing GB' and KD' we immediately find KG.

31. *To find a relation between the breadth, depth, slopes of sides, and area of transverse section of a ditch, the form of the section being a trapezoid.*



Let $ACDB$ be a transverse section of a ditch,

A = area of the section,

$AB = x$ = breadth of the top,

$EC = d$ = depth of the ditch;

and suppose that AC slopes m feet in 1,

and BD......... n...........,

which means that AC falls m feet vertically for one foot of horizontal distance, and BD n feet in the same distance.

Now the section being a trapezoid,

$$\text{Area} = \frac{1}{2}(AB + CD)\,EC.$$

$$\text{But } CD = AB - AE - FB$$

$$= AB - \frac{EC}{m} - \frac{FD}{n};$$

$$\because \frac{AE}{EC} = \frac{1}{m}, \quad \frac{FB}{FD} = \frac{1}{n};$$

$$\therefore CD = x - \frac{d}{m} - \frac{d}{n}$$

$$= x - \frac{m+n}{mn}d;$$

$$\therefore A = \frac{d}{2}\left\{x + x - \frac{m+n}{mn}d\right\},$$

$$\text{or } A = xd - \frac{m+n}{2mn}d^2.$$

This equation between A, x, d, m and n, will determine any one of these quantities when the other four are given.

In applying it to any particular case it will be simpler to substitute the values of the four given quantities before proceeding with the solution. If the slopes of the sides be the same, i. e. if $n = m$, the equation becomes

$$A = xd - \frac{d^2}{m}.$$

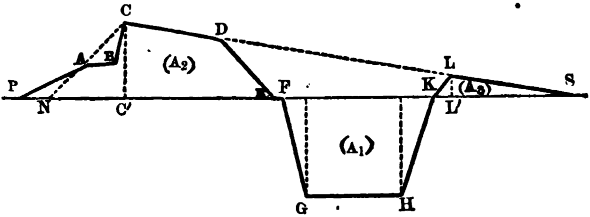
If the ditch be excavated until the sloping sides meet each other, $CD = 0$,

$$\text{and } \therefore\ x=\frac{m+n}{mn}d,$$

$$\text{and } A=\frac{1}{2}AB\times EC=\frac{m+n}{2mn}d^2,$$

$$\text{and if also the slopes be equal, } A=\frac{d^2}{m}.$$

32. *To find the breadth which must be given to a ditch, so that the area of its transverse section may be equal to that of the breastwork and glacis together, subject to the condition that the superior slope of the parapet shall be in the same plane with the glacis,—the depth, and slopes of the sides of the ditch being given, and also the dimensions of the profile of the breastwork.*



Let PS be the plane of site, or original surface of the ground, supposed horizontal; and let $PABCDEFGHKLS$ be the profile of the breastwork, ditch, and glacis.

Then, by the condition respecting the superior slope of the parapet, $CDLS$ will be a straight line.

Let x = breadth of the ditch at top = FK,

d = depth ....,

m in 1 the slope of FG,

n in 1 .... HK,

p in 1 .... KL.

Also, let A_1 = area of transverse section of ditch,

A_2 = ... breastwork,

A_3 = ... glacis.

Then we have, by the first condition,

$$A_1 = A_2 + A_3 \;\ldots\ldots\ldots\ldots\ldots(1).$$

Now A_2 is known, because the dimensions of the breastwork are given, but A_1 and A_3 evidently depend upon x, the quantity we have to find.

We must therefore express the values of A_1 and A_3 *in terms of* x, and substitute them in equation (1), which will then become an equation for determining x.

For A_1, we have, from Art. 31,

$$A_1 = xd - \frac{m+n}{2mn} d^2.$$

For A_3, we have to find the area of the triangle KLS. Now since the dimensions of the breastwork are given, the line CD is fixed, and therefore the point S is determinate. Draw CC', LL' perpendicular to the plane of site PS, and CN parallel to KL.

Let $\qquad FS = a, \quad FC' = b, \quad CC' = h.$

Now $\qquad A_3 = \frac{1}{2} KS \times LL',$

$$KS = FS - FK = a - x,$$

and $\dfrac{LL'}{C'} = \dfrac{LS}{CS} = \dfrac{KS}{NS}$, by similar triangles;

$$\therefore\; LL' = CC' \frac{KS}{NS} = h \frac{a-x}{a+b+\frac{h}{p}},$$

$$\because\; NS = FS + FC' + C'N, \text{ and } C'N = p \,.\, CC';$$

$$\therefore\; A_3 = \frac{h}{2} \frac{(a-x)^2}{a+b+\frac{h}{p}}.$$

Substituting these values of A_1 and A_3 in equation (1), we have

$$xd - \frac{m+n}{2mn} d^2 = A_2 + \frac{h}{2} \frac{(a-x)^2}{a+b+\frac{h}{p}}.$$

This is a quadratic equation in x, and will give real values for x whenever the form and dimensions of the breastwork, and the depth of the ditch, are such as to render the problem possible.

The equation will also serve to determine the depth of the ditch when the breadth of the top is given, the other conditions remaining the same; for, when x is given, the equation becomes a quadratic in d.

In each case there will be *two* values of the unknown quantity, but the nature of the problem will show which value is admissible. In solving the equation, either for x or d, it will be simpler to substitute the values of all the known quantities before proceeding to the solution.

33. The equation last found will require some modification in practice to allow for the fact that the quantity of earth which is taken out of an excavation will be more than sufficient to fill it again.

The proportion of this excess will depend upon the nature of the earth. For our present purpose we may represent the excess per cubic foot by e, so that one cubic foot of earth will, after excavation, occupy a space $1+e$. Now the equation

$$A_1 = A_2 + A_3,$$

is an equation between the *areas* of the sections of parapet, ditch, and glacis, but here we have to consider the *volumes* of these parts.

If we suppose the common length of the parapet, ditch, and glacis to be l, the equation between their volumes will be

$$lA_1 = lA_2 + lA_3,$$

as will be seen in the following Chapter, and it is the quantity lA_1 which must be replaced by $lA_1(1+e)$ to allow for the excess of earth; therefore we have, instead of equation (1),

$$lA_1(1+e) = lA_2 + lA_3,$$

$$\text{or,}\quad A_1(1+e) = A_2 + A_3.$$

This will require a smaller value of A_1 than the geometrical equivalent of A_2 and A_3, and therefore, with the same breadth, a smaller value of the depth, or, with the same depth, a smaller value of the breadth.

The resulting equation will, of course, be

$$(1+e)\left\{xd - \frac{m+n}{2mn}d^2\right\} = A_2 + \frac{h}{2}\,\frac{(a-x)^2}{a+b+\frac{h}{p}},$$

and we should, in practice, make use of this equation instead of that of the preceding Article.

There is no other method by which x or d can be found with the same accuracy and facility.

CHAPTER V.

VOLUMES OF SOLIDS CONTAINED BY PLANE RECTILINEAR FIGURES.

34. In measuring the volume of a solid figure, whether it be bounded by plane or curve surfaces, we have to determine the ratio which the volume of the given solid bears to the volume of some other solid which is chosen arbitrarily, and is called the unit of volume.

Definition of a Unit of Volume. The volume contained by three planes at right angles to each other, each of which is a unit of area, is called a unit of volume.

Hence the unit of volume is a cube, each edge of which is a unit of length.

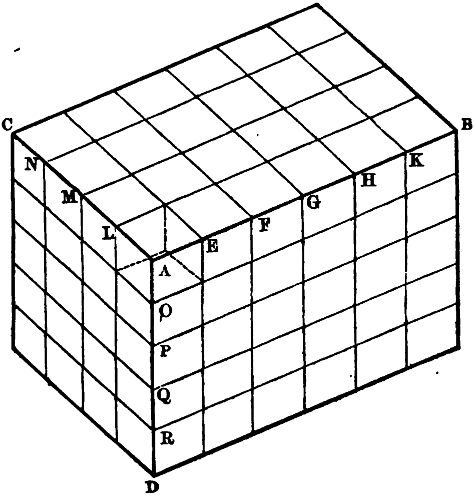
Thus, if the unit of length be an inch, the unit of volume will be that contained by three square inches at right angles to each other, and will be called a cubic inch; and, similarly, if the unit of length be a foot, or a yard, the unit of volume will be called a cubic foot, or a cubic yard, respectively.

35. *To find an expression for the volume of a rectangular parallelepiped in terms of three adjacent edges.*

Def. A parallelepiped is a solid figure contained by six quadrilaterals every opposite two of which are parallel.

It will follow from this definition that all the quadrilaterals are parallelograms.

When these parallelograms are rectangles, the solid is called a rectangular parallelepiped.



Let *AB*, *AC*, *AD* be three adjacent edges of a rectangular parallelepiped.

Let a be the number of units of length in *AB*,
… b ………………………………………… *AC*,
… c ………………………………………… *AD*,

and V the number of units of volume in the parallelepiped; we shall prove that

$$V = abc.$$

There will be three cases :—

(1) When a, b, c are *integers*.

(2) When a, b, c are, one or more, *fractional*.

(3) When a, b, c are *incommensurable*.

Case (1). When a, b, c are ***integers.***

Divide AB into a equal parts in the points E, F, G, H, K,
AC ...... b .... L, M, N,
and AD ...... c .... O, P, Q, R.*

Through the points E, F, G, H, K draw planes parallel to CAD,
through the points L, M, N draw planes parallel to DAB,
and .... O, P, Q, R .... BAC.

These planes divide the solid into a number of cubes, each of which is a unit of volume. The number of these cubes in any horizontal layer will manifestly be the same as the number of units of area in the rectangle BAC, or ab; and since the number of horizontal layers equals c, the whole number of cubes in the parallelepiped will be ab taken c times, or abc.

$$\therefore\ V = abc.$$

Case (2). When a, b, c are, one or more, *fractional.*

We shall suppose them all three fractional.

Let α represent the unit of length, so that

$$AB = a.\alpha,\quad AC = b.\alpha,\quad AD = c.\alpha;$$

and suppose

$$a = p + \frac{h}{m},$$

$$b = q + \frac{k}{n},$$

$$c = r + \frac{l}{s},$$

where p, q, r are positive integers, and $\frac{h}{m}$, $\frac{k}{n}$, $\frac{l}{s}$ proper fractions in their lowest terms.

Now take another unit of length β, less than α, and such that

$$\alpha = mns\,\beta.$$

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[*] In the figure the lines AB, AC, AD are divided into 6, 4 and 5 equal parts respectively, for the sake of convenience, as explained in Art. 3, Chap. I.

Then
$$a.\alpha=\left(p+\frac{h}{m}\right)mns\beta=(pm+h)\,ns\beta,$$

$$b.\alpha=\left(q+\frac{k}{n}\right)mns\beta=(qn+k)\,ms\beta,$$

$$c.\alpha=\left(r+\frac{l}{s}\right)mns\beta=(rs+l)\,mn\beta;$$

and these latter expressions for $a.\alpha$, $b.\alpha$, and $c.\alpha$ being *whole numbers* of the unit of length β, their product will, by Case (1), be the number of units of volume in the parallelepiped, each of which is the cube upon the unit of length β, and will be represented by β^3.

$\therefore$ Volume $=(pm+h)\,(qn+k)\,(rs+l)\,m^2n^2s^2$ units of volume β^3,

$$=\frac{pm+h}{m}\cdot\frac{qn+k}{n}\cdot\frac{rs+l}{s}\,m^3n^3s^3 \text{ units of volume } \beta^3,$$

$$=\left(p+\frac{h}{m}\right)\left(q+\frac{k}{n}\right)\left(r+\frac{l}{s}\right)\text{units of volume } \alpha^3,$$

because a unit of volume corresponding to the unit of length α contains $m^3n^3s^3$ units of volume corresponding to the unit of length β, by Case (1).

Hence, as in the first case,

$$V=abc.$$

Case (3). When a, b, c are *incommensurable.*

In this case, although we cannot find any unit which will measure AB, AC and AD, yet, as we may take a unit *as small as we please*, we can always find three lines *which shall be commensurable*, and which shall differ from AB, AC and AD by quantities *less than any that can be assigned;* and their product will be the volume of a parallelepiped differing from the given parallelepiped by a quantity *less than any assignable.*

Hence, in this case also, the number of units of volume in the parallelepiped must be the product of the numbers of units of length in three adjacent edges. Therefore, in all cases, we have the formula

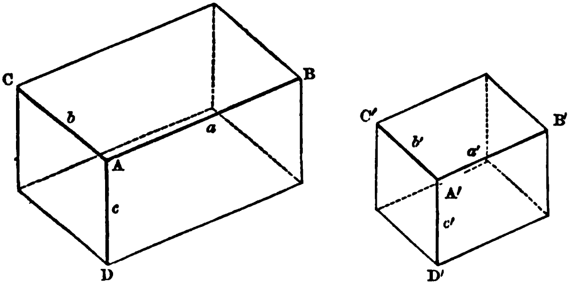
$$V=abc.$$

Cor. If $b=c=a$, the solid will be a cube, and its volume will be a^3.

We shall now give another proof of this proposition, derived from pure Geometry, similar to that which we have already given in the case of areas.

36. *To find an expression for the volume of a rectangular parallelepiped.*

In Euclid's *Elements of Geometry*, Book XI. Proposition *D*, it is proved that, "Solid parallelepipeds which have equal solid angles have to one another the ratio which is compounded of the ratios of their sides," and therefore rectangular parallelepipeds, which have equal solid angles, have to one another this same ratio.



Let V and V' represent the volumes of two rectangular parallelepipeds, and let their edges be represented by a, b, c, and a', b', c', respectively.

Then by the above-mentioned proposition,

$$\frac{V}{V'}=\frac{a}{a'}\times\frac{b}{b'}\times\frac{c}{c'}=\frac{abc}{a'b'c'};$$

$$\therefore\ V=V'\frac{abc}{a'b'c'}\ \ldots\ldots\ldots\ldots(1).$$

This equation is the foundation of the measurement of volumes, but, as in the case of areas, before we can apply it in practice we must define a standard of volume by which all other volumes may be measured.

The usual convention with regard to units of volume is, that the volume which is contained by three units of area at right angles to each other shall be taken for a unit of volume. Thus a unit of volume will be a cube, each edge of which is a unit of length. Now if in the right hand figure we suppose a', b', c' to be, each of them, a unit of length, V' will become a unit of volume, and equation (1) will become

$$V = abc \ldots\ldots\ldots\ldots\ldots (2).$$

This equation shows that the number of units of volume in a rectangular parallelepiped is that number which is the product of the numbers of units of length in three adjacent edges.

The proof just given includes every case which can occur, whether the edges a, b, c be represented by whole numbers of the unit of length, or by fractional or incommensurable numbers, since it is derived from pure Geometry which is independent of all units of measurement. The student is recommended to compare this proof with that given in Art. 4, Chap. I. and to apply the remarks in that Article to the present proof.

Cor. The expression abc may be written

$$ab \times c, \text{ or } bc \times a, \text{ or } ac \times b;$$

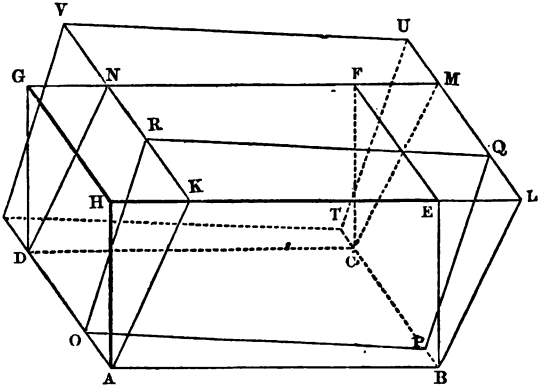
which shows that the volume of a rectangular parallelepiped is equal to the area of any one face multiplied by the length of an edge perpendicular to that face. Hence, if A be the area of the base, and h the altitude of the solid,

$$V = Ah.$$

37. *To find an expression for the volume of any parallelepiped.*

In Euclid's *Elements*, Book XI. Prop. 31, it is proved that, "Solid parallelepipeds which have equal bases and equal altitudes are equal in volume,"—but this proposition may be established more briefly as follows.

Let *ABCD . EFGH* represent a rectangular parallelepiped. Through *AD* and *BC* draw parallel planes *AKND*, *BLMC*, and complete the solid *EBL . FCM*. This solid is equal in all respects to the solid *HAK . GDN*; for, it has the same edges, and the same solid angles contained by the same plane angles, and we may conceive it to be



applied to the solid *HAK . GDN* so that the points *B*, *C*, *F*, *E* shall coincide, respectively, with the points *A*, *D*, *G*, *H*, when the points *L*, *M* will also coincide with *K*, *N*, and the two solids will coincide, and will therefore be equal in volume. Therefore the parallelepiped *ABCD . KLMN* is equal to the rectangular parallelepiped *ABCD . EFGH*.

In the same manner, if we draw any line *OP*, make *OS* equal to *AD*, draw *ST* parallel to *OP*, and through *OP* and *ST* draw parallel planes *OPQR*, *STUV*, we may show that the parallelepiped thus formed will be equal in volume to the parallelepiped *ABCD . KLMN*, and therefore equal to the rectangular parallelepiped. Now the parallelepiped *OPTS . RQUV* and the rectangular parallelepiped have equal bases *OPTS* and *ABCD*, and equal altitudes, and they are proved to be equal in volume; and in the same manner we may show that *any* parallelepiped will be equal in volume to a rectangular parallelepiped which has an equal base and an equal altitude. Hence, if V be the volume of any parallelepiped, A the area of the base, and h the altitude,

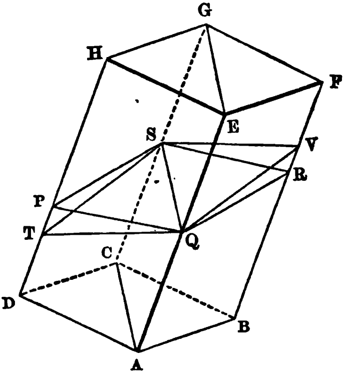
$$V = Ah.$$

38. *To find an expression for the volume of a prism*—

(1) When the base is a triangle.

(2) When the base is any plane polygon.

For definition of prism, see Art. 12, Chap. I.



Case (1). *When the base is a triangle.*

Let $ABCD.EFGH$ represent a parallelepiped. The plane which passes through the opposite diagonals AC and EG divides the solid into two equal parts. This is proved in Euclid's *Elements*, Book XI. Prop. 28; but it may easily be shown as follows.

Let Q and S be the middle points of AE and CG; and through SQ draw any two planes $PQRS$, $TQVS$.

Then it may easily be proved that the two triangles SPQ and QRS are equal in all respects. Hence the solid $SPQT$ may be applied to the solid $QRSV$ so that the points S, P, Q may coincide, respectively, with the points

Q, R, S, when it will be manifest that the two solids will be symmetrical about the plane QRS, and equal to each other.

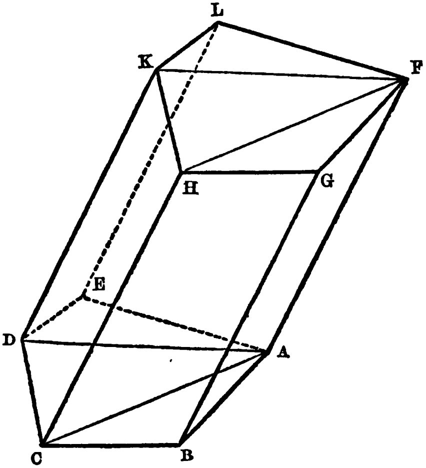
For the same reason, any two opposite solids which have SQ for a common edge, will be equal in volume; and the sum of the one set which make up the prism $ABC.EFG$ will be equal to the sum of the other set which make up the prism $ACD.EGH$, and therefore these two prisms will be equal in volume.

Therefore the volume of either prism will be half the volume of the parallelepiped; also, the base of either prism is half the base of the parallelepiped, and their altitudes are the same.

Hence, if A be the area of the base of a triangular prism, h the altitude, and V the volume,

$$V = Ah.$$

Case (2). *When the base is any plane polygon.*



Let $ABCDE.FGHKL$ represent a prism the base of which is the polygon $ABCDE$.

By drawing planes through any one edge, as AF, and through all the other edges excepting the two adjacent to AF, one on either side of it, we can divide the prism into a number of *triangular* prisms, all of which have a common altitude, (viz. the perpendicular distance between the planes which form the base and top of the prism,) and their bases will be the triangles BAC, CAD, DAE.

Hence if A_1, A_2, A_3 and A be the areas of these three triangles and of the polygon, respectively,

V_1, V_2, V_3 and V the volumes of the three triangular prisms and of the whole prism, and h the common altitude,

$$V_1 = A_1 h\,; \quad V_2 = A_2 h\,; \quad V_3 = A_3 h\,;$$

$$\therefore\ V_1 + V_2 + V_3 = (A_1 + A_2 + A_3)\,h,$$

$$\text{or,}\ V = Ah.$$

In the same manner, whatever be the number of sides of the polygon, it may be proved that the number of units of volume in the prism is equal to the number of units of area in the base multiplied by the number of units of length in the altitude.

Cor. 1. If the prism be divided into two parts by a plane perpendicular to the parallel edges, and the upper part be applied to the lower so that the face $FGHKL$ may coincide with the face $ABCDE$, F coinciding with A, G with B, and so on; the solid thus formed will be a *right* prism, having for its base the transverse section, and for its altitude the length, of the given prism; and its volume will, of course, be the same.

Hence if A' be the area of the transverse section, and l the length of the prism,

$$V = A'l.$$

Cor. 2. Since $V = A'l = Ah$, $\therefore\ \dfrac{A'}{A} = \dfrac{h}{l}$.

Now $\dfrac{h}{l}$ is manifestly the cosine of the angle between the planes of the transverse section and base, since h is the perpendicular from F upon the plane of the base, and $l = FA$, and is perpendicular to the plane of the transverse section.

Hence if A be the area of a rectilinear figure upon any plane, A' the area of its *projection* upon another plane making an angle θ with the former, then

$$A' = A \cos \theta.$$

This result will also be true for a curvilinear area, as will be seen in a subsequent Chapter.

39. *To find an expression for the volume of a pyramid—*

(1) When the base is a triangle.

(2) When the base is any plane polygon.

For definition of pyramid, see Art. 15, Chap. I.

This proposition presents peculiar difficulties.

If we *assume*, as is sometimes done, that "pyramids upon equal bases and of equal altitudes are equal in volume," the difficulties are evaded; but, in a Treatise like the present, in which every proposition is strictly proved, or immediately deduced from Euclid's *Elements*, such an assumption is inadmissible.

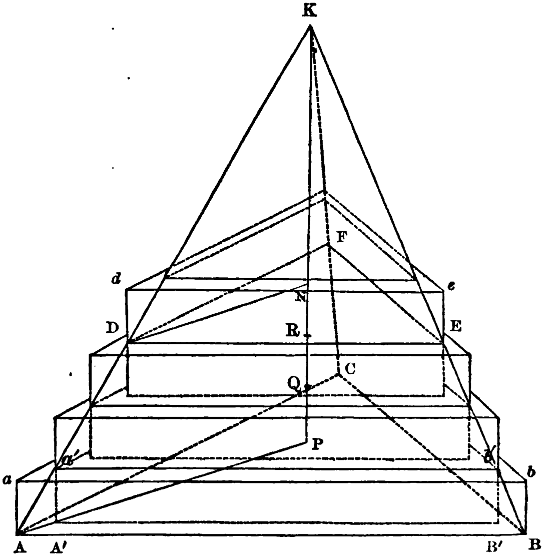
Again, if we refer to Euclid's *Elements of Geometry* to *prove* the above-mentioned proposition, we shall find it necessary to employ Propositions 3, 4, 5 and 6, of Book XII. to establish it. These propositions, which are long and difficult, are not usually read; to avoid them we shall here employ, for the first time, the "Method of Limits."

This method will be fully developed and applied in the subsequent Chapters, but it will be no disadvantage to the student to see some application of it at this stage of his progress.

Case (1). *When the base is a triangle.*

Let K be the vertex, and ABC the base, of a triangular pyramid. Draw KP perpendicular to the base, intersecting it in P, and divide KP into any number of equal parts, say n, in the points Q, R, N, &c.

Through each of these points of division draw planes parallel to the base *ABC*, and complete the prisms *ABab*, &c., as shown in the figure, and also the prisms *A′B′a′b′*, &c., as shown by the dotted lines.



It will be manifest that the pyramid will be always less than the *complete* outer pile of prisms, and always greater than the *complete* inner pile; and that the difference between the pyramid and either pile will be less than the difference between the piles themselves.

But this difference will be the prism *ABab*; for, the *lowest* prism of the inner pile is equal, in all respects, to the *second* prism of the outer pile, and the *second* prism of the inner pile to the *third* prism of the outer pile, and so on, to the highest of each; and therefore the inner pile of prisms is equal, in all respects, to the outer pile, excepting the prism at the base. But, when the number of prisms is increased indefinitely, and the altitude of each indefinitely diminished, this prism will become less than any assignable magnitude; therefore the difference between the two piles of prisms, and therefore, *a fortiori*, the difference between the pyramid and either pile will become less than any assignable magnitude. Hence the pyramid is ultimately equal in volume to either pile of prisms.

We shall now find the volume of a pyramid by first finding an expression for the volume of the outer pile of prisms, and then finding what this expression becomes when n is indefinitely great.

Let A be the area of the base ABC, h the altitude of pyramid, $=KP$, and therefore each of the parts into which KP is divided $=\frac{h}{n}$; also, let KN be the first r of these parts, measured from K, so that $KN=r\frac{h}{n}$.

Now the prism $DEFde =$ (area of $\triangle\ DEF) \times \frac{h}{n}$, but since DEF is a section of the pyramid made by a plane parallel to the base, it will be similar to the base;

$$\therefore\ \triangle DEF : \triangle ABC :: DE^2 : AB^2, \text{ Euclid, VI. 19.}$$

$$:: KD^2 : KA^2, \text{ by similar triangles,}$$

$$:: KN^2 : KP^2, \text{ ....};$$

$$i.e.\ \triangle DEF : A :: r^2\frac{h^2}{n^2} : h^2;$$

$$\therefore\ \triangle DEF = A\frac{r^2}{n^2};$$

$$\therefore \text{ volume of prism } DEFde = Ah\frac{r^2}{n^3}$$

Hence, the volume of the outer pile of prisms will be found by giving to r, in this expression, all values from 1 to n, both inclusive, and taking the sum of the resulting quantities;

$$\therefore \text{ volume of outer pile} = \frac{Ah}{n^3}(1^2+2^2+3^2+\ldots\ldots+n^2).$$

The sum of the series $1^2+2^2+3^2+\dots\dots+n^2$, may easily be shown to be $\frac{1}{6}n(n+1)(2n+1)$. See Art. 67, Chap. VI.

$$\therefore \text{ volume of outer pile} = \frac{Ah}{6n^2}(n+1)(2n+1)$$

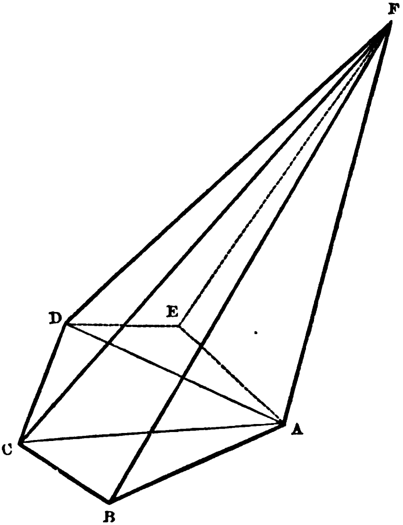
$$= \frac{Ah}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right),$$

Now, when n is indefinitely great, this expression will give the volume of the pyramid; but in that case, $\frac{1}{n}$, $\frac{1}{2n}$ vanish compared with unity;

$$\therefore \text{ volume of pyramid} = \frac{Ah}{3}.$$

Hence a pyramid is *one-third* of a prism of equal base and altitude; and pyramids of equal bases and equal altitudes are equal in volume.

Case (2). *When the base is any plane polygon.*



Let $F.ABCDE$ represent a pyramid the base of which is the polygon $ABCDE$.

Draw the planes FAC, FAD, which will divide the pyramid into three pyramids on *triangular* bases BAC, CAD, DAE.

Let A_1, A_2, A_3 and A be the areas of these triangles and of the polygon, respectively, V_1, V_2, V_3 and V the volumes of the three triangular pyramids and of the given pyramid, and h the common altitude, viz. the perpendicular from the vertex F upon the plane of the base.

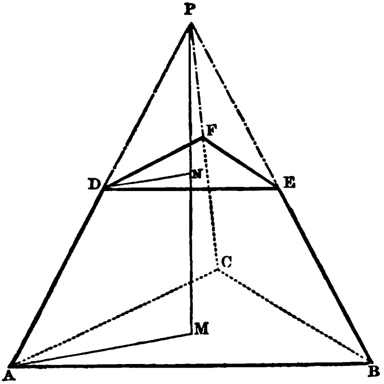
Then $V_1 = \frac{1}{3}A_1 h$; $V_2 = \frac{1}{3}A_2 h$; $V_3 = \frac{1}{3}A_3 h$;

$$\therefore V_1 + V_2 + V_3 = \tfrac{1}{3}(A_1 + A_2 + A_3)h,$$

$$\text{or, } V = \tfrac{1}{3}Ah.$$

In the same manner this formula may be proved, whatever may be the number of sides of the base of the pyramid.

40. *To find an expression for the volume of a frustum of a pyramid, made by a plane parallel to the base.*



Let $ABC.DEF$ be a frustum of a pyramid, made by a plane DEF parallel to the base ABC.

Let P be the vertex of the complete pyramid.

Draw PM perpendicular to the base ABC, intersecting the plane DEF in the point N, and join AM and DN.

Let A_1 be the area of the triangle ABC,

A_2 ………………………………… DEF,

MN, the altitude of the frustum, $=h$,

$PN=x$, and let $V=$ volume of the frustum.

Now the frustum $=$ pyramid $P.ABC-$ pyramid $P.DEF$;

$$\therefore\ V=\tfrac{1}{3}A_1(x+h)-\tfrac{1}{3}A_2x$$

$$=\tfrac{1}{3}\{A_1h+(A_1-A_2)x\};$$

and from this expression the quantity x must be eliminated because it is unknown.

Now the triangles ABC and DEF are similar;

$$\therefore\ A_1 : A_2 :: AB^2 : DE^2. \quad \text{(Euclid, VI. 19.)}$$

But $$AB : DE :: PA : PD,$$

$$:: PM : PN,$$

$$:: x+h : x;$$

$$\therefore\ A_1 : A_2 :: (x+h)^2 : x^2;$$

$$\therefore\ \frac{\sqrt{A_1}}{\sqrt{A_2}}=\frac{x+h}{x}=1+\frac{h}{x},$$

$$\therefore\ \frac{\sqrt{A_1}-\sqrt{A_2}}{\sqrt{A_2}}=\frac{h}{x},$$

$$\therefore\ x(A_1-A_2)=h\sqrt{A_2}\{\sqrt{A_1}+\sqrt{A_2}\}.$$

Substituting this value of $x(A_1-A_2)$ in the expression already found for V, we have

$$V=\frac{h}{3}\{A_1+\sqrt{A_1A_2}+A_2\}.$$

In this proof the figure used is a triangular pyramid, but the same proof will apply to a pyramid the base of which is any plane polygon, for in such case, A_1 and A_2, instead of being the areas of two similar triangles, would be the areas of two similar polygons of which AB and DE would be homologous sides, and we should have in this case also,

$$A_1 : A_2 :: AB^2 : DE^2. \quad \text{(Euclid, VI. 20.)}$$

Hence, whatever may be the form of the pyramid,

$$V = \frac{h}{3}\{A_1 + \sqrt{A_1 A_2} + A_2\}.$$

41. *To find an expression for the volume of a frustum of a triangular prism, made by two planes inclined to each other at any angle.*

Let $ABC \, . \, DEF$ be a frustum of a triangular prism, made by the two planes ABC and DEF, inclined to each other at any angle.

Draw the planes GBH, KEL perpendicular to BE, one of the edges of the prism.

Now the volume of the frustum is equal to the volume of the prism $GBH \, . \, KEL$ together with the volumes of the two solids, the one, $KELFD$, above, and the other, $GBHCA$, below this prism.

Join KF and DL.

Let A be the area of the triangle GBH, or the area of the transverse section of the frustum.

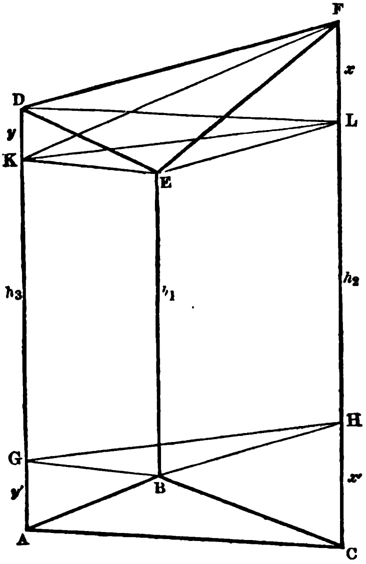
Let $EB = h_1$, $FC = h_2$, $DA = h_3$,

$FL = x$, $HC = x'$, $DK = y$, $AG = y'$,

and let V be the volume of the frustum.

Now the solid $KELFD$ is made up of the two triangular pyramids $KEF \, . \, L$ and $KEF \, . \, D$, of which $KEF \, . \, L$ is the same thing as $KEL \, . \, F$, and $KEF \, . \, D$ is the same thing as $KED \, . \, F$.

Also $KED \,.\, F = KED \,.\, L$, because they are pyramids on the same base KED and of equal altitudes, and $KED \,.\, L$ is the same thing as $KEL \,.\, D$.



Hence the solid $KELFD$ is equal to the two pyramids $KEL \,.\, F$ and $KEL \,.\, D$;

$$i.e. \text{ the solid } KELFD = \tfrac{1}{3}Ax + \tfrac{1}{3}Ay;$$

$$\text{similarly, the solid } GBHCA = \tfrac{1}{3}Ax' + \tfrac{1}{3}Ay',$$

$$\text{and the prism } GBH \,.\, KEL = Ah_1;$$

$$\therefore\ V = Ah_1 + \tfrac{1}{3}A(x + y + x' + y')$$

$$= A\{h_1 + \tfrac{1}{3}(x + x') + \tfrac{1}{3}(y + y')\};$$

$$\text{but } x + x' = h_2 - h_1, \text{ and } y + y' = h_3 - h_1,$$

$$\therefore \quad V = A\left\{h_1 + \frac{h_2 - h_1}{3} + \frac{h_3 - h_1}{3}\right\},$$

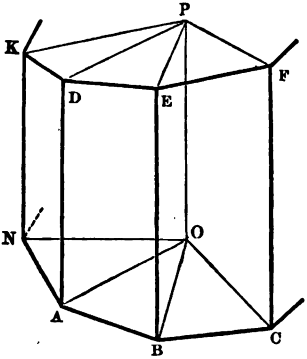
$$\text{or, } V = A\,\frac{h_1 + h_2 + h_3}{3}.$$

This formula shows that the volume of the frustum is equal to the area of the transverse section multiplied by the *mean length* of the three parallel edges.

The formula, of course, includes the simpler case in which one end of the frustum is itself perpendicular to the three parallel edges, or, as it would then be called, a frustum of a right prism.

42. *To find an expression for the volume of a frustum of a prism, the base of which is any regular polygon.*

Let $ABC...N$ be a portion of a regular polygon of n sides which forms the base of a prism; and suppose the frustum to be made by a plane $DEF...K$ inclined at any angle to the plane $ABCN$.



From O, the centre of the polygonal base, draw OP perpendicular to the plane of the base, to intersect the plane $DEF...K$ in P.

Join $OA, OB, OC, \ldots\ldots ON$; $PD, PE, PF, \ldots\ldots PK$.

Let V = volume of the whole frustum,

V_1 = ... frustum on the base AOB,

V_2 = ... BOC,

&c. = &c...,

V_n = ... NOA,

$OP=h$, $AD=h_1$, $BE=h_2$, $CF=h_3$, &c. = &c., $NK=h_n$.

Also, let A = area of the base of the whole frustum, and therefore $\frac{A}{n}$ = area of each of the triangles AOB, BOC, ...NOA.

Then
$$3V_1=\frac{A}{n}(h+h_1+h_2),$$

$$3V_2=\frac{A}{n}(h+h_2+h_3),$$

$$\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$3V_n=\frac{A}{n}(h+h_n+h_1).$$

Also
$$V_1+V_2+\ldots\ldots+V_n=V;$$

$$\therefore\ 3V=\frac{A}{n}(nh+2h_1+2h_2+\ldots\ldots+2h_n).$$

Now it may very easily be shown that

$$nh=h_1+h_2+\ldots\ldots+h_n,$$

whether the number of sides of the polygon be even or odd.

Making this substitution for nh, we have

$$3V=\frac{A}{n}3(h_1+h_2+\ldots\ldots+h_n),$$

$$\text{or,}\quad V=A\,\frac{h_1+h_2+\ldots\ldots+h_n}{n}.$$

This formula shows that the volume of the frustum is equal to the area of the transverse section multiplied by the *mean length* of all the parallel edges.

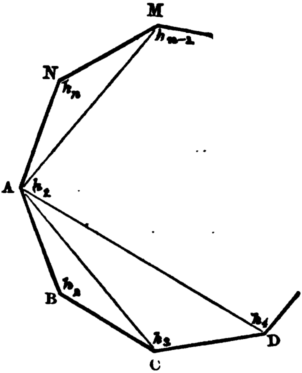
It will easily be seen that the formula is as general as that of the preceding Article, in which the ends of the frustum are *both* inclined to the parallel edges, or to the axis of the prism.

43. The formula of the preceding Article may also be proved, as follows, by a method which is sometimes useful, and which we shall now give.

Let *ABCD* ...... *MN* be a regular polygon of *n* sides, forming the base, or transverse section, of a frustum of a prism.

Join *AC*, *AD*, ... *AM*.

Let h_1, h_2, h_3, h_4, ... h_{n-1}, h_n be the lengths of the edges of the frustum, which pass through the points *A*, *B*, *C*, *D*, ... *M*, *N*, respectively.



And let A_1, A_2, ... A_{n-2} be the areas of the triangles *BAC*, *CAD*, ... *MAN*,—the number of these triangles being *less by two* than the number of sides of the polygon—and *A* the area of the section.

Let V_1, V_2, ... V_{n-2} be the volumes of the solids which stand upon the bases A_1, A_2, ... A_{n-2}, respectively, and V the volume of the whole solid.

Then we have

$$3V_1 = A_1(h_1 + h_2 + h_3),$$
$$3V_2 = A_2(h_1 + h_3 + h_4),$$
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots,$$
$$3V_{n-2} = A_{n-2}(h_1 + h_{n-1} + h_n).$$

Also $$V_1 + V_2 + \ldots\ldots V_{n-2} = V;$$

$$\therefore\ 3V = A_1(h_1 + h_2 + h_3) + A_2(h_1 + h_3 + h_4) + \ldots$$
$$+ A_{n-2}(h_1 + h_{n-1} + h_n);$$

and we shall obtain a similar equation by repeating the foregoing process with each of the points B, C, D, ...M, N in succession; therefore also,

$$3V = A_1(h_2 + h_3 + h_4) + A_2(h_2 + h_4 + h_5) + \ldots\ldots\ldots$$
$$+ A_{n-2}(h_2 + h_n + h_1),$$
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$3V = A_1(h_n + h_1 + h_2) + A_2(h_n + h_2 + h_3) + \ldots$$
$$+ A_{n-2}(h_n + h_{n-2} + h_{n-1});$$

adding these n equations together, we find

$$3nV = (A_1 + A_2 + \ldots + A_{n-2})\,(3h_1 + 3h_2 + \ldots\ldots + 3h_n),$$

$$\text{or,}\quad nV = A(h_1 + h_2 + \ldots\ldots + h_n),$$

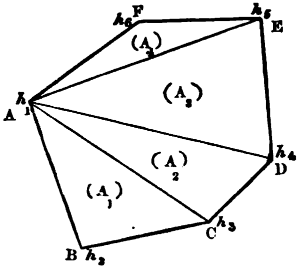
since $A = A_1 + A_2 + \ldots\ldots\ldots A_{n-2}$;

$$\therefore\ V = A\,\frac{h_1 + h_2 + \ldots + h_n}{n}.$$

44. *To find an expression for the volume of a frustum of a prism, the transverse section of which is any irregular polygon.*

Let *ABCDEF* represent the transverse section of such a prism; and let it be divided into triangles by straight lines drawn through any one angular point, as A.

Let A_1, A_2, A_3, A_4 be the areas of these triangles as shown in the figure,



h_1, h_2, h_3, h_4, h_5, h_6 the lengths of the edges through the points *A*, *B*, *C*, *D*, *E*, *F*, respectively.

V_1, V_2, V_3, V_4 the volumes of the frusta of the triangular prisms of which A_1, A_2, A_3, A_4 are, respectively, the transverse sections, and let *V* be the whole volume.

Then
$$3V_1 = A_1(h_1 + h_2 + h_3),$$
$$3V_2 = A_2(h_1 + h_3 + h_4),$$
$$3V_3 = A_3(h_1 + h_4 + h_5),$$
$$3V_4 = A_4(h_1 + h_5 + h_6);$$

hence by addition,

$$3V = h_1(A_1 + A_2 + A_3 + A_4) + h_2A_1 + h_3(A_1 + A_2)$$
$$+ h_4(A_2 + A_3) + h_5(A_3 + A_4) + h_6A_4.$$

This is expressed in a rule as follows, the truth of which will be manifest from the figure:

"Multiply the length of each edge of the frustum by the sum of the areas of all the triangles in the transverse section which have an angular point in that edge. The sum of the products will be three times the volume."

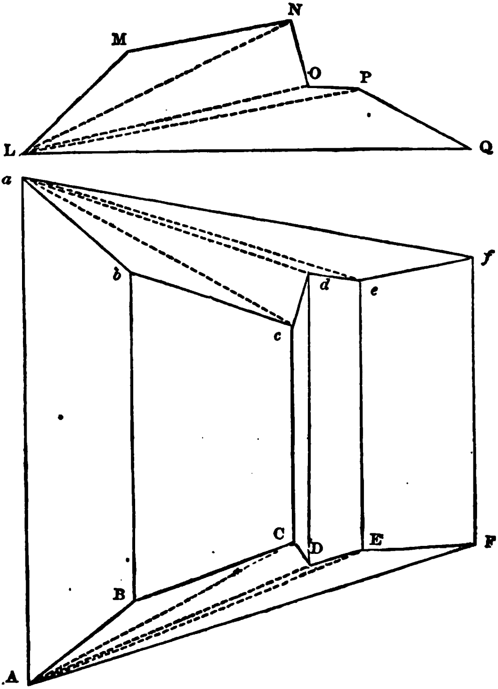
It is manifest that this rule will apply to a frustum of a prism of any number of sides.

We may conveniently express the above formula by an abridged notation, as follows:

$$3V = \Sigma . Ah,$$

where Σ stands for "the sum of all the quantities of the form," and A represents the sum of the areas of all the triangles which have an angular point in the edge, the length of which is h.

45. *To find the volume of any portion of a parapet.*



The lower figure represents a portion of a parapet contained between two planes *ABCDEF* and *abcdef*, which may be vertical planes, or planes inclined to the vertical, and to the direction of the parapet, at any angles.

The upper figure represents a transverse section of the parapet; and it will now be manifest that the solid under consideration is a frustum of a prism the transverse section of which is an irregular polygon, and therefore the formula of the preceding Article will be sufficient for the determination of its volume.

It should be observed that in every case Aa, Bb, Cc, Dd, Ee, Ff are *lines on the plan*, and can therefore be measured.

Let their lengths be $x_1, x_2, x_3, x_4, x_5, x_6$, respectively, and let A_1, A_2, A_3, A_4 be the areas of the triangles MLN, NLO, OLP, PLQ, into which the section is divided by lines drawn through the point L, and which can be easily calculated as explained in Art. 29, Ch. IV. Also, let V be the volume of the solid; then, by the preceding Article,

$$3V = \Sigma . Ax,$$

or, if we develope this formula,

$$3V = x_1(A_1 + A_2 + A_3 + A_4) + x_2A_1 + x_3(A_1 + A_2) + x_4(A_2 + A_3) + x_5(A_3 + A_4) + x_6A_4 .$$

The formula

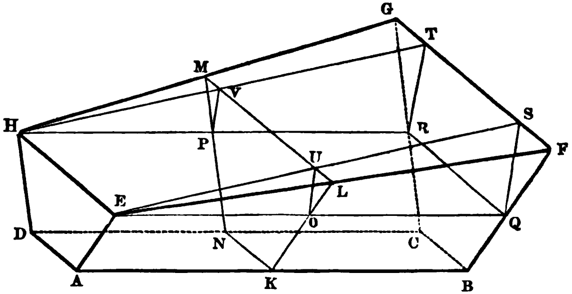
$$3V = \Sigma . Ax$$

will also determine the volume of any portion of a rampart or other similar work, care being taken to divide the transverse section into triangles all of which have a common vertex in one of the *lines on the plan;* for, if the section be divided into triangles in any other manner, the formula will be applicable only when the bounding planes ABF, abf are *vertical*.

46. *To find an expression for the volume of a prismoid.*

Fig. 1 represents a Prismoid,—which is a solid contained by a rectangle $ABCD$ and the trapezoids $ABFE$, $DCGH$, $ADHE$ and $BCGF$, and of these, the last two have their planes perpendicular to the edge *AB*. The remaining side *EFGH* will necessarily be a trapezoid also.

Fig. 1.



The planes *ABFE*, *DCGH* may be inclined to the base *ABCD* at any angles, not necessarily the same.

Fig. 2 represents an elevation of the prismoid, taken at the smaller end, and is the only figure necessary in practice.

A familiar example of a prismoid is a railway cutting, where the roadway is ***straight*** and ***horizontal***, and the original surface of the ground an ***inclined plane*** running in the ***same direction*** as the rails.

We shall now investigate the following simple formula for the volume of a prismoid, viz.

$$6V = l\{A_1 + A_2 + 4A\},$$

where l = length of the prismoid = *AB*,

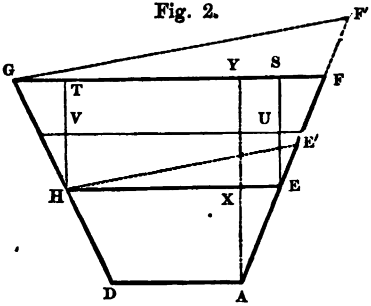
A_1 = area of smaller end *ADHE*,

A_2 = ... larger....... *BCGF*,

A = ... mean, or mid-section *KLMN*, made by a plane parallel to the ends, passing through *K*, the middle point of *AB*,

and V = volume of the solid.

Fig. 2.



Through *EH*, Fig. 1, draw the plane *EQRH* parallel to the base *ABCD*, and through *EQ* and *HR* draw planes *EQS*, *HRT* perpendicular to the plane *EQRH*.

These three planes divide the prismoid into four solids, viz., the prism *BH*, the triangular prism *TQH*, and the two pyramids *E.SQF* and *H.TRG*, all of which have a common length, or altitude, equal to *AB*.

Let A_3 = area of the rectangle *QT*,

A_4 = ... triangle *SQF*,

A_5 = ... *TRG*,

then, since A_1 is the area of the trapezoid *BR*,

$A_1 l$ is the volume of the prism *BH*,

$\frac{1}{2}A_3 l$ .... prism *TQH*,

$\frac{1}{3}A_4 l$ .... pyramid *E.SQF*,

and $\frac{1}{3}A_5 l$ .... pyramid *H.TRG*,

and these four solids make up the prismoid;

$$\therefore\ V = A_1 l + \tfrac{1}{2}A_3 l + \tfrac{1}{3}A_4 l + \tfrac{1}{3}A_5 l,$$

or, $$6V = l\{6A_1 + 3A_3 + 2A_4 + 2A_5\}.$$

Now $$A_2 = A_1 + A_3 + A_4 + A_5,$$

$$A = A_1 + \tfrac{1}{2}A_3 + \tfrac{1}{4}A_4 + \tfrac{1}{4}A_5;$$

{$\because OP=QR$, $OU=\frac{1}{2}QS$, $OL=\frac{1}{2}QF$, and $PM=\frac{1}{2}RG$; and the triangles UOL, SQF are *similar*, as also are the triangles VPM and TRG},*

and $$\therefore\ 4A=4A_1+2A_3+A_4+A_5.$$

Hence $$A_1+A_2+4A=6A_1+3A_3+2A_4+2A_5.$$

Therefore, by substitution in the above expression for $6V$, we have
$$6V=l\{A_1+A_2+4A\}.$$

For the application of this formula in practice, we only require to know the breadths AD, EH, FG, the perpendiculars AX, AY, (see fig. 2) and the length AB, of the prismoid.

Let these lines be represented by a, b, c, p_1, p_2 and l, respectively.

Then
$$A_1=\tfrac{1}{2}(a+b)p_1,$$
$$A_2=\tfrac{1}{2}(a+c)p_2,$$
$$A=\tfrac{1}{2}\{a+\tfrac{1}{2}(b+c)\}\frac{p_1+p_2}{2},$$
$$\therefore\ 4A=\tfrac{1}{2}(2a+b+c)(p_1+p_2).$$

By substituting these values of A_1, A_2, and $4A$ in the formula
$$6V=l\{A_1+A_2+4A\},$$
we readily obtain the volume V.

47. The formula which we have proved in the last Article will be true *however small* may be the breadth AD; therefore it will be true when AD vanishes. In that case the section becomes a triangle, and the solid a frustum of a triangular pyramid. Hence it follows that if the upper surface of the prismoid be *inclined* to the base, as represented by $GF'E'H$, fig. 2, (in which case the transverse sections are no longer trapezoids), the same formula will hold; for, it will hold for each of the parts into which the solid is divided by the plane $EFGH$, and therefore for the whole solid.

---

* In Fig. 1, the line OP has been accidentally omitted.

48. We may now show that the formula which we have already obtained for the volume of a frustum of a triangular pyramid, Art. 40, is identical with that which we have just deduced for the same solid, by supposing the section of the prismoid a triangle instead of a trapezoid.

We have from Art. 40,

$$V=\frac{h}{3}(A_1+\sqrt{A_1A_2}+A_2);$$

$$\therefore 6V=h(A_1+A_1+2\sqrt{A_1A_2}+A_2+A_2)$$

$$=h\{A_1+(\sqrt{A_1}+\sqrt{A_2})^2+A_2\}.$$

Now if A be the area of the mean section of the frustum.

k the length of the side parallel to AB,

then $$\frac{A_1}{A}=\frac{AB^2}{k^2},\ \frac{A_2}{A}=\frac{DE^2}{k^2};$$

$$\therefore \sqrt{A_1}+\sqrt{A_2}=\frac{AB+DE}{k}\sqrt{A}:$$

but $$AB+DE=2k;$$

$$\therefore \sqrt{A_1}+\sqrt{A_2}=2\sqrt{A};\quad \therefore (\sqrt{A_1}+\sqrt{A_2})^2=4A.$$

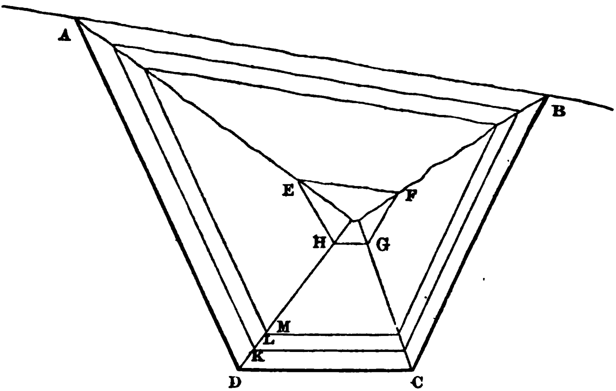
Hence $$6V=h\{A_1+A_2+4A\},$$

the same formula as before.

49. The formula $6V=l\{A_1+A_2+4A\}$ will hold for all solids the ends of which are polygons of the same number of sides, with their planes parallel, and the remaining faces trapezoids; for all such solids can be divided into prismoids of the *same length*, and since the areas of the ends of the solid are equal to the sums of the areas of the ends of the prismoids, and the area of the mean section is equal to the sum of the areas of the mean sections of the prismoids, the same formula which will be true of each prismoid will be true of their sum, *i.e.* of the whole solid: it will also hold when any of the trapezoids become triangles by the vanishing of their smaller ends, in which case the ends of the solid will *not* have the same number of sides.

Such solids are called "Prismoidal Solids," and the general formula which is applicable to them is called the "Prismoidal Formula."

50. Application of the Prismoidal Formula to determine the volume of earth in a railway cutting, or embankment, or other similar work.



The figure represents a perspective view of a railway cutting, contained between two transverse sections *ABCD* and *EFGH*,—*ABFE* representing the original surface of the ground.

Divide *DH* into an *even* number of equal parts, say $2n$, in the points *K*, *L*, *M*, &c., and through each of these points draw planes parallel to *ABCD*.

Let the areas of these planes, taken in order, be $A_1, A_2, A_3, \ldots\ldots A_{2n+1}$,—$A_1$ being the area of *ABCD*, and A_{2n+1} that of *EFGH*; and let each of the distances *DK*, *KL*, *LM*, &c.$=h$, so that $DH=2nh$.

Now the earth between any two *alternate* sections is prismoidal in form, if we neglect the inequalities in the surface of the ground between such sections, and the intermediate section will be the mean section of this prismoidal solid. Hence

$$6\,\text{Vol. between } A_1 \text{ and } A_3 = 2h\,(A_1 + A_3 + 4A_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\; A_3 \ldots\ldots A_5 = 2h\,(A_3 + A_5 + 4A_4),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\; A_5 \ldots\ldots A_7 = 2h\,(A_5 + A_7 + 4A_6),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots\; A_{2n-1} \ldots A_{2n+1} = 2h\,(A_{2n-1} + A_{2n+1} + 4A_{2n}).$$

$$\therefore\ 6 \text{ whole volume} = 2h\,\{A_1 + A_{2n+1} + 2\,(A_3 + A_5 + \ldots A_{2n-1}) + 4\,(A_2 + A_4 + \ldots A_{2n})\};$$

$$\therefore\ 3V = h\,\{A_1 + A_{2n+1} + 2\,(A_3 + A_5 + \ldots\ldots + A_{2n-1}) + 4\,(A_2 + A_4 + \ldots\ldots + A_{2n})\}.$$

Hence we have this rule:—

"Divide the length of the cutting into an *even* number of equal parts, and take the areas of the transverse sections at the points of division.

"Add together the areas of the extreme sections, twice the sum of the areas of all the intermediate sections of an *odd* order, and four times the sum of the areas of all the intermediate sections of an *even* order, and multiply by the distance between two consecutive sections: the product will be three times the volume."

The Formula of this Article may be expressed symbolically as follows:—

$$3V = h\,\{\Sigma\,.\,A_{\text{extreme}} + 2\Sigma\,.\,A_{\text{odd}} + 4\,\Sigma\,.\,A_{\text{even}}\};$$

where $\Sigma\,.\,A_{\text{extreme}}$ represents the sum of the areas of the extreme sections, and $\Sigma\,.\,A_{\text{odd}}$, $\Sigma\,.\,A_{\text{even}}$ the sums of the areas of all the intermediate sections of an *odd* and *even* order respectively.

It will be observed that the formula is only approximate, on account of the inequalities in the surface of the ground having been neglected; but the nearer the sections are taken to each other, or the greater the number of the parts into which the length of the cutting is divided, the more accurate will be the result.

51. We may also apply the same method, and the same formula, to determine the volume of a cutting or embankment when made upon a curve, the planes of the transverse sections being supposed to converge to the centre of curvature. In this case the volume of earth between alternate sections will not be accurately prismoidal, even when we neglect the inequalities of the surface, because no two sections are *parallel;* but if we take for h the *mean* distance between two consecutive sections,—which will, in general, be the distance measured along the centre line of the roadway,—the errors will be very small.

In all cases, the closer the sections are taken to each other the more accurate will be the result.

In a subsequent Chapter we shall investigate an extension of the Prismoidal Formula to certain solids bounded by curve surfaces, and it will then be seen that the approximation to the volume of a solid, such as that considered in Art. 50, is much closer than would appear from the assumption upon which the formula of that Article is founded; and that this formula does include, with considerable accuracy, the inequalities in the surface of the ground which were neglected in establishing it.

CHAPTER VI.

ON THE METHOD OF LIMITS.

52. *Definition of a Limit.*

The limit of a variable quantity or magnitude is that quantity or magnitude to which it may be made continually to approximate; but to which it never becomes actually equal, although the difference may be made less than any assignable.

Let X represent a variable quantity,

A a constant or invariable quantity,

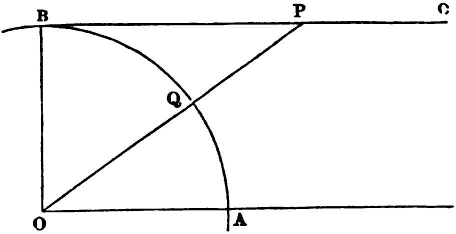
then *if* X can be made continually to approximate to A, and *if* the difference can be made less than any assignable, though not absolutely zero, A is the limit of X.

As an illustration, let X represent the sum of n terms of the Geometric Series $1, \frac{1}{2}, \frac{1}{4}$, &c. Then, by the theory of geometric series, we know that

$$X = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

Now, by continually increasing n, we can make X continually approximate to 2, and the difference between 2 and X, or $\frac{1}{2^{n-1}}$, can be made less than any assignable. Hence 2 is the limit of X; or 2 is the limit of the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}$, when n is indefinitely great.

Again, let OA, OB be two radii of a circle at right angles to each other, and let BC be a tangent to the circle at B, and therefore parallel to OA. Suppose two points



P and Q to start from B at the same instant, the one, P, moving along the line BC, and the other, Q, along the arc BA, but in such a manner that the straight line PQ shall always pass through O. Then it will be evident that as P moves along BC, the path BQ, travelled by Q, will continually approximate to the quadrant BA, and that the difference QA can be made less than any assignable. Moreover it is evident that the difference QA can never be zero, since the point P can never leave the line BC, and QA can only vanish when OP coincides with OA.

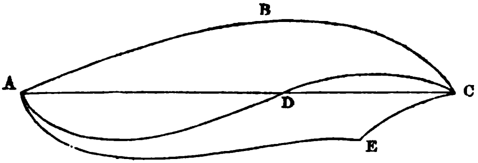
Hence the quadrant BA is that magnitude to which BQ can be made continually to approximate, but to which it never becomes actually equal, although the difference QA can be made less than any assignable. Therefore BA is the limit of BQ.

Obs. One magnitude is frequently said to be *ultimately equal* to another: all that is meant by this is, that the difference between the two magnitudes can be made less than any assignable magnitude, but not actually zero.

53. *Definition of a Curve.*

A curve is a line traced by a moving point which is continually changing the direction of its motion.

When the change in direction is gradual, and nowhere abrupt between two points, the curvature is said to be *continued* between those points. And, when the curve is wholly convex to the line joining any two points, the curvature is said to be in the *same direction* between those points.

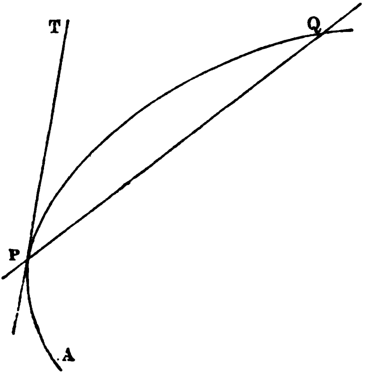


Thus the curve *ABC* is *continued*, and in the *same direction* between the points *A* and *C*; the curve *ADC* is *continued*, but *not* in the *same direction* between these points; and the curve *AEC* is neither continued nor in the same direction between the points *A* and *C*. In the last case there is a sudden change in the direction of the motion of the generating point at *E*.

54. *Definition of the tangent to a curve.*

The tangent to a curve at any point is the straight line in which the generating point *would* move, if it *ceased* to change the direction of its motion at that point.

We shall find it convenient to use another definition of the tangent, which is as follows:—



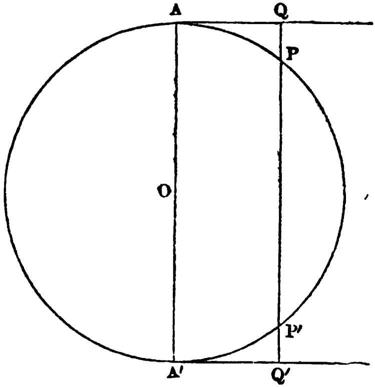
Let APQ be any curve, the curvature of which is *continued* through the point P, and in the *same direction* between two near points P and Q. Draw the chord PQ, and let the point Q move along the curve continually towards P, so that the distance PQ may be less than any assignable. Then the line PQ will continually approximate towards some *determinate position* PT. This line PT, which is the limiting position of PQ, is the tangent to the curve at the point P.

55. Lemma I.

Quantities, and the ratios of quantities, which, in any finite time, tend continually to equality, and which, before the end of that time, approach nearer to each other than by any assigned difference, become ultimately equal.

If not, let them become ultimately unequal, and let their ultimate difference be D. Therefore, (since they tend *continually* to equality), they cannot approach nearer to each other than by the difference D, which is contrary to the hypothesis. Therefore, they do not become ultimately unequal, that is, they become ultimately equal.

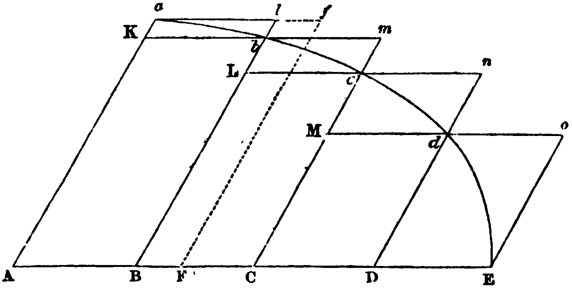
As an illustration of Lemma I, let AA' be a diameter of a circle, PP' a chord parallel to AA'; AQ, $A'Q'$ tangents at A and A', (and therefore perpendicular to AA'), intersecting PP' produced, in Q and Q'.



Now, as the line QPQ' moves, parallel to AA', up to AA', PP' and QQ' tend *continually* to equality, and before PP' arrives at AA', their difference ($2QP$) can be made less than any assignable. Hence PP' and QQ' become ultimately equal.

56. Lemma II.

If, in any figure AacE, *bounded by the straight lines* Aa, AE *and the curve* acE, *any number of parallelograms* Ab, Bc, Cd, *&c. be inscribed upon equal bases* AB, BC, CD, *&c., and having the sides* Bb, Cc, Dd, *&c. parallel to the side* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, *&c. be completed; then, if the breadth of these parallelograms be diminished, and the number increased indefinitely, the ultimate ratios which the inscribed figure* AKbLcMdD, *the circumscribed figure* AalbmcndoE, *and the curvilinear figure* AabcdE, *have to one another, are ratios of equality.*



For the difference of the inscribed and circumscribed figures is the sum of the parallelograms Kl, Lm, Mn, Do, that is, (since the bases of all are equal), a parallelogram whose base is Kb, that of one of them, and altitude the sum of their altitudes, that is, the parallelogram $ABla$. But this parallelogram, since its breadth is indefinitely diminished, becomes less than any assignable magnitude, therefore (by Lemma I.) the inscribed and circumscribed figures, and, *a fortiori*, the intermediate curvilinear figure, become ultimately equal.

57. Lemma III.

The same ultimate ratios are also ratios of equality, when the breadths of the parallelograms AB, BC, CD, *&c. are unequal, and all are diminished indefinitely.*

For, let AF be equal to the greatest breadth, and the parallelogram $FAaf$ be completed. This parallelogram will be greater than the difference between the inscribed and circumscribed figures. But, when its breadth is diminished indefinitely, it will become less than any assignable parallelogram.

Cor. 1. Hence the ultimate sum of the evanescent parallelograms coincides with the curvilinear figure.

Cor. 2. And, *a fortiori*, the rectilinear figure which is bounded by the chords of the evanescent arcs ab, bc, cd, &c. ultimately coincides with the curvilinear figure.

Cor. 3. As also the rectilinear figure which is bounded by the tangents of the same arcs.

Cor. 4. And these ultimate figures, with respect to their perimeters acE, are not rectilinear, but curvilinear limits of rectilinear figures.

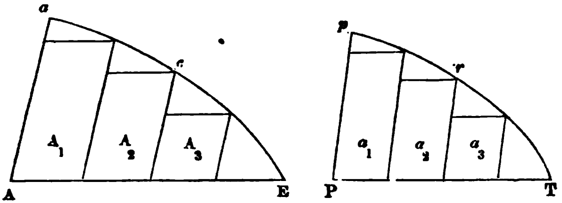
The curve acE is the limit of the boundary formed by the chords, or the tangents.

58. If, in the figure of Lemma II, Aa be at right angles to AE, the parallelograms will be rectangles; and, if we then suppose the whole figure to revolve round AE fixed, the curvilinear area will generate a solid, (called a solid of revolution), and the inscribed and circumscribed figures will each generate a series of cylinders; and it will be manifest from the Lemma that the ultimate ratios of either series of cylinders and of the solid of revolution

will be ratios of equality. The straight line *AE*, about which the figure revolves, is called the *axis* of the solid of revolution.

59. Lemma IV.

If in two figures AacE, PprT *there be inscribed* (*as in Lemma II.*) *two series of parallelograms, the number in each series being the same, and if, when the breadths are diminished indefinitely, the ultimate ratios of the parallelograms in one figure to the parallelograms in the other be the same, each to each; then, the two figures* AacE, PprT *are to one another in that same ratio.*



For, as the parallelograms are each to each, so, componendo, is the sum of all to the sum of all, and so the figure *AacE* to the figure *PprT*, for, by Lemma III, the former figure is to the former sum, and the latter figure to the latter sum in a ratio of equality.

The following proof will assist the student in understanding this important Lemma.

Let $A_1, A_2, A_3 \ldots, a_1, a_2, a_3 \ldots$ be the areas of the two series of parallelograms inscribed in the two figures, and let

$$\frac{A_1}{a_1} = m + x_1, \quad \frac{A_2}{a_2} = m + x_2, \quad \frac{A_3}{a_3} = m + x_3, \quad \text{\&c.} = \text{\&c.},$$

x_1, x_2, x_3, &c., being quantities which vanish in the limit, when the breadths of the parallelograms are diminished indefinitely, so that, according to the hypothesis, the limits of $\frac{A_1}{a_1}$, $\frac{A_2}{a_2}$, $\frac{A_3}{a_3}$, &c., are each equal to m.

Now $A_1 = ma_1 + x_1a_1$, $A_2 = ma_2 + x_2a_2$,

$$A_3 = ma_3 + x_3a_3, \text{ \&c.} = \text{\&c.};$$

$$\therefore A_1 + A_2 + A_3 + \text{\&c.} = m(a_1 + a_2 + a_3 + \text{\&c.})$$

$$+ x_1a_1 + x_2a_2 + x_3a_3 + \text{\&c.};$$

$$\therefore \frac{A_1 + A_2 + A_3 + \text{\&c.}}{a_1 + a_2 + a_3 + \text{\&c.}} = m + \frac{x_1a_1 + x_2a_2 + x_3a_3 + \text{\&c.}}{a_1 + a_2 + a_3 + \text{\&c.}}.$$

Now if x be always the greatest of the variable quantities x_1, x_2, x_3, &c., this last ratio will be less than

$$\frac{xa_1 + xa_2 + xa_3 + \dots}{a_1 + a_2 + a_3 + \dots},$$

i. e. less than x,—a quantity which vanishes in the limit.

Hence the limit of $\frac{A_1 + A_2 + A_3 + \dots}{a_1 + a_2 + a_3 + \dots} = m$,

$$\text{i.e. } \frac{\text{area of figure } AacE}{\text{area of figure } PprT} = m.$$

COR. Hence, if two quantities of any kind whatever be divided into any the same number of parts, and these parts, when their number is increased and magnitude diminished indefinitely, have the same given ratio each to each, viz. the first to the first, the second to the second, and so on in order; the whole quantities will be to one another in the same given ratio. For, if in the figures of this Lemma, the parallelograms be taken, each to each, in the same ratio as the parts, the sums of the parts will be always as the sums of the parallelograms: and, therefore, when the number of the parts and parallelograms is increased, and their magnitude diminished indefinitely, in the ultimate ratio of parallelogram to parallelogram, that is, (by hypothesis), in the ultimate ratio of part to part.

60. *Definition of similar curvilinear figures.*

One curvilinear figure is said to be similar to another, when, *any* rectilinear figure being inscribed in the one, a *similar* rectilinear figure can be inscribed in the other.

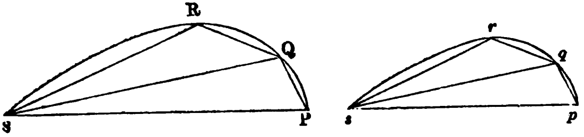
Hence two similar curvilinear figures are the *limits* of two similar rectilinear figures, when the number of sides is increased indefinitely, and the magnitude of each indefinitely diminished.

61. Lemma V.

All the homologous sides of similar figures are proportional, whether curvilinear or rectilinear, and their areas are in the duplicate ratio of the sides.

Newton does not give any demonstration of this Lemma. It follows of necessity from the definition of similar curvilinear figures, that they are the *limits* of similar rectilinear figures, and whatever is true of these rectilinear figures is true of their limits.

62. We shall now deduce a simple test of the similarity of curvilinear figures.



Let SPR, spr be two similar curvilinear figures, and $SPQR$, $spqr$ similar rectilinear figures inscribed in them. Join SQ, sq; then PSQ, psq will be similar triangles, and the angles PSQ, psq will be equal;

$$\therefore SQ : SP :: sq : sp,$$

$$\text{or } SQ : sq :: SP : sp.$$

Now since this is true of *any* similar rectilinear figures, it will be true for *any* points Q, q, provided the angles PSQ, psq are equal.

Hence the proportion "$SQ : sq :: SP : sp$" will be a test of the similarity of any two curves.

Thus we may show that all circles are similar. For, if we suppose the above figures to represent semicircles of which SP, sp are diameters,

$$\therefore SQ = SP \cos PSQ, \quad sq = sp \cos psq,$$

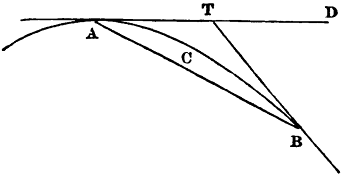
$$\therefore SQ : sq :: SP : sp,$$

since the angles PSQ, psq are equal.

Or we may reason thus:—If we take two concentric circles, and describe *any polygon whatever* in the outer circle, and from the angular points draw radii to the common centre of the circles; then, if we join the points, in order, where these radii intersect the inner circle, we shall form a polygon which will be *similar* to the first polygon. Hence the circles are similar.

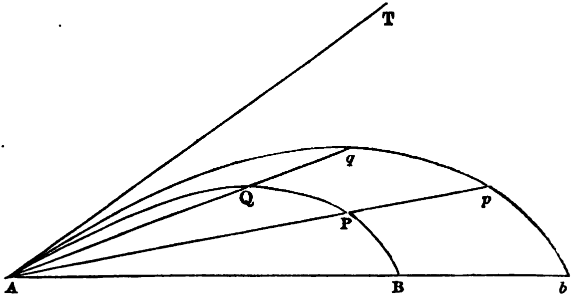
63. Lemma VI.

If any arc ACB, *given in position, be subtended by a chord* AB, *and if at any point* A, *in the middle of continued curvature, it be touched by the straight line* AD *produced in both directions; then, if the points* A, B *approach one another and ultimately coincide; the angle* BAD, *contained by the chord and the tangent, will diminish indefinitely and ultimately vanish.*



Draw BT the tangent at B; then, since the point A is in the middle of continued curvature, when B moves up to A, the tangents BT and AT will ultimately coincide; therefore the angle BTD will diminish indefinitely and ultimately vanish, and therefore, *a fortiori*, the angle BAT, which is less than BTD, will diminish indefinitely and ultimately vanish.

64. *Similar conterminous arcs, which have their chords coincident, have a common tangent.*



Let APB, Apb be two similar conterminous arcs, having their chords AB, Ab coincident.

Through A draw *any* two straight lines APp, AQq. Then, since the curves are similar,

$$AP : Ap :: AB : Ab,$$

$$AQ : Aq :: AB : Ab; \quad \text{Art. 62.}$$

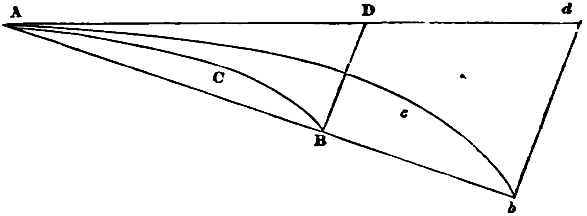
$$\therefore AQ : Aq :: AP : Ap.$$

Hence the arcs AQP, Aqp are also similar. Now, if the point P move along the curve up to and ultimately coincide with A, the two arcs AP, Ap, being always similar, will vanish together, and the straight line APp will assume the *same* ultimate position for either curve, i.e. the two curves have a common tangent.

DEF. The *subtense* of an arc is a straight line drawn from one extremity of it, to meet, at a finite angle, the tangent at the other extremity.

65. Lemma VII.

If any arc* ACB, *given in position, be subtended by the chord* AB, *and at the point* A, *in the middle of continued curvature, the tangent* AD *be drawn, and the subtense* BD; *then, when* B *approaches to* A *and ultimately coincides with it, the ultimate ratio of the arc, the chord, and the tangent to one another is a ratio of equality.

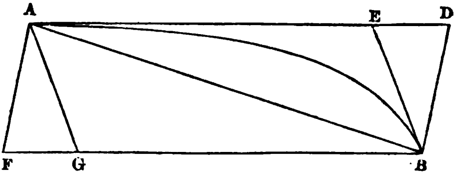


For, as the point B approaches to the point A, let AB, AD be supposed always to be produced to points b, d at a finite distance from A, and bd be drawn parallel to the subtense BD, and let the arc Acb be always similar to the arc ACB, and therefore having AD for its tangent at A.

Now, when the points B, A coincide, the angle bAd, by the preceding Lemma, will vanish, and therefore the straight lines Ab, Ad, which are always finite, and the intermediate arc Acb will ultimately coincide, and will therefore be equal. Hence, also, the straight lines AB, AD and the intermediate arc ACB, which are always proportional to them, and which vanish together, will have an ultimate ratio of equality to each other.

Cor. 1. Hence, if through B, BF be drawn parallel to the tangent, always cutting any straight line AF, passing through A, in F, this line BF will have ultimately to the vanishing arc ACB a ratio of equality, since, if the parallelogram $AFBD$ be completed, it has always a ratio of equality to AD.

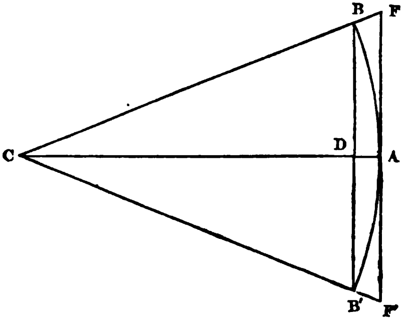
Cor. 2. And if, through B and A, any number of straight lines BE, BD, AF, AG be drawn cutting the



tangent AD and the straight line BF parallel to it; the ultimate ratio of all the abscissæ AD, AE, BF, BG and of the chord and arc AB to one another will be a ratio of equality.

Cor. 3. And, therefore, in every argument concerning ultimate ratios, all these lines may be used indifferently one for the other.

66. *To find the limiting values of* $\frac{\sin\theta}{\theta}$ *and* $\frac{\tan\theta}{\theta}$ *when* θ *is diminished indefinitely,* θ *representing the circular measure of an angle less than a right angle.*



Let BAB' be an arc of a circle, BB' its chord, FF'

the tangent parallel to the chord, terminated by the radii CB, CB' produced to meet it in F and F' respectively.

Draw CA bisecting the angle BCB', and therefore perpendicular to FF'.

Let R be the radius of the circle, and θ the circular measure of the angle BCA;

$$\therefore \theta = \frac{BA}{R}, \quad \sin\theta = \frac{BD}{R}, \quad \tan\theta = \frac{FA}{R}.$$

Now, by Lemma VII, the chord BB', the arc BAB', and the tangent FF' have an ultimate ratio of equality to each other when the angle BCB' is indefinitely diminished;

$$\therefore BD, \ BA \text{ and } FA \text{ are ultimately equal;}$$

$\therefore \frac{BD}{R}, \frac{BA}{R}$ and $\frac{FA}{R}$, or $\sin\theta$, θ, and $\tan\theta$, are ultimately equal;

$$\therefore \frac{\sin\theta}{\theta}, \ 1, \text{ and } \frac{\tan\theta}{\theta} \text{ are ultimately equal;}$$

or the limiting values of $\frac{\sin\theta}{\theta}$ and $\frac{\tan\theta}{\theta}$ are each equal to unity.

If we suppose the angle to be the n^{th} part of four right angles, its circular measure will be expressed by $\frac{2\pi}{n}$, and $\frac{2\pi}{n}$ will be diminished indefinitely when n is indefinitely increased;

$$\therefore \text{the limiting value of } \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = 1, \text{ when } n = \infty.$$

It is in this form that we shall generally make use of this limiting value in the following pages.

67. In the application of the method of limits to determine the lengths of curve lines, the areas of surfaces, and the volumes of solids, we shall require to know the limiting value of the expression

$$\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}},$$

when n is indefinitely increased, p representing any positive integer.

This limit may be found by summing the series and taking the limit of the resulting expression. But instead of adopting this course, which would require the actual summation of the series in all cases, we shall here introduce a Theorem which will determine the sums of all series similar to that above when n is *finite*, as well as the limiting values of all expressions of the above form when n is indefinitely increased.

Let S_1 denote the series $1+2+\ldots\ldots+n$,

S_2 .... $1^2+2^2+\ldots\ldots+n^2$,

and, generally, S_p .... $1^p+2^p+\ldots\ldots+n^p$.

By the Binomial Theorem, p being any positive integer,

$$(n-1)^p=n^p-pn^{p-1}+p\frac{p-1}{2}n^{p-2}-\ldots\mp pn\pm 1,$$

the terms being alternately positive and negative, and the last term being ± 1, according as p is even or odd.

Therefore

$$n^p-(n-1)^p=pn^{p-1}-p\frac{p-1}{2}n^{p-2}+\ldots\pm pn\mp 1,$$

where the last term will now be ∓ 1, according as p is even or odd.

Now by giving to n all the values $1, 2, 3, \ldots n$, successively, we obtain the following equations:—

$$1^p-0^p=p.1^{p-1}-p\frac{p-1}{2}.1^{p-2}+\ldots\pm p.1\mp 1,$$

$$2^p-1^p=p.2^{p-1}-p\frac{p-1}{2}.2^{p-2}+\ldots\pm p.2\mp 1,$$

$$3^p-2^p=p.3^{p-1}-p\frac{p-1}{2}.3^{p-2}+\ldots\pm p.3\mp 1,$$

$$\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$n^p-(n-1)^p=p.n^{p-1}-p\frac{p-1}{2}.n^{p-2}+\ldots\pm p.n\mp 1.$$

Adding these equations together, and substituting $S_1, S_2, \ldots S_{p-1}$ for the sums of the series which they respectively represent, we have

$$n^p=pS_{p-1}-p\frac{p-1}{2}S_{p-2}+\ldots\pm pS_1\mp n,$$

$$\text{or}\quad n^p-pS_{p-1}+p\frac{p-1}{2}S_{p-2}-\ldots\mp pS_1\pm n=0,$$

the number of terms being $p+1$.

From this Theorem all the series $S_1, S_2, \ldots S_{p-1}$ may be found in succession, beginning with S_1.

To find S_1, let $p=2$;

$$\therefore\ n^2-2S_1+n=0;$$

$$\therefore\ S_1=\frac{n^2+n}{2}=\frac{n(n+1)}{2}.$$

Again, to find the limit of $\frac{S_1}{n^2}$ or $\frac{1+2+\ldots+n}{n^2}$, when n is indefinitely increased, we have

$$\frac{S_1}{n^2}=\frac{1}{2}\left(1+\frac{1}{n}\right),$$

and, by sufficiently increasing n, $\frac{1}{n}$ can be made less than any assignable quantity, and $\therefore$ vanishes compared with unity.

Hence the limit of $\frac{S_1}{n^2}=\frac{1}{2}$.

To find S_2, let $p=3$;

$$\therefore\ n^3-3S_2+3S_1-n=0;$$

$$\therefore\ S_2=\frac{1}{3}(n^3+3S_1-n)$$

$$=\frac{1}{3}\left\{n^3+\frac{3n}{2}(n+1)-n\right\}$$

$$=\frac{n}{6}\{2(n^2-1)+3(n+1)\}$$

$$=\frac{n}{6}(n+1)(2n+1).$$

To find the limit of $\frac{S_2}{n^3}$ when n is indefinitely increased, we do not *require* to know the value of S_2 in terms of n, for, dividing the above equation by n^3, we have

$$1-3\frac{S_2}{n^3}+3\frac{S_1}{n^3}-\frac{1}{n^2}=0,$$

and passing to the limit,

$$1-3 \text{ limit of } \frac{S_2}{n^3}=0;$$

—for the limit of $\frac{S_1}{n^2}$ having been proved to be finite, that of $\frac{S_1}{n^3}$ will be zero, and also the limit of $\frac{1}{n^2}$ is zero—

$$\therefore \text{ the limit of } \frac{S_2}{n^3}=\frac{1}{3}.$$

Similarly, we may find S_3 from the equation

$$n^4-4S_3+6S_2-4S_1+n=0;$$

and dividing by n^4, and passing to the limit, we have

$$1-4 \text{ limit of } \frac{S_3}{n^4}=0;$$

$$\text{and } \therefore \text{ the limit of } \frac{S_3}{n^4}=\frac{1}{4}.$$

In the same manner, if we divide the general equation by n^p, we have

$$1-p\,\frac{S_{p-1}}{n^p}+p\,\frac{p-1}{2}\,\frac{S_{p-2}}{n^p}-\ldots\mp p\,\frac{S_1}{n^p}\pm\frac{1}{n^{p-1}}=0,$$

in which it will now be manifest that all the terms after the second will vanish in the limit;

$$\therefore \text{ the limit of } \frac{S_{p-1}}{n^p}=\frac{1}{p},$$

$$\text{and the limit of } \frac{S_p}{n^{p+1}}=\frac{1}{p+1};$$

or $\frac{1}{p+1}$ is the limiting value of $\frac{1^p+2^p+\ldots+n^p}{n^{p+1}}$, when n is indefinitely increased, p representing any positive integer.

This result is also true when p is a positive fraction.

COR. Any *finite* number of terms may be omitted from the numerator without affecting the limit. For, supposing the first r terms to be omitted,

$$\text{the limit of } \frac{(r+1)^p+(r+2)^p+\ldots+n^p}{n^{p+1}}$$

$$=\text{the limit of } \left\{\frac{1^p+2^p+\ldots+n^p}{n^{p+1}}-\frac{1^p+2^p+\ldots+r^p}{n^{p+1}}\right\}$$

$=\frac{1}{p+1}$, since the limit of the second fraction is *zero*, its numerator being always *finite*.

Thus the limits of

$$\frac{2^p+3^p+\ldots+n^p}{n^{p+1}}, \text{ and } \frac{1^p+2^p+\ldots+(n-1)^p}{n^{p+1}},$$

when n is indefinitely increased, are each equal to $\frac{1}{p+1}$.

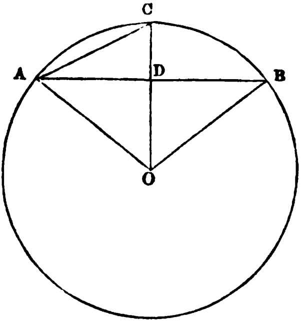
CHAPTER VII.

AREAS OF PLANE CURVILINEAR FIGURES.

68. In this chapter we shall consider the areas of plane curvilinear figures when the curvilinear boundaries are described according to known laws. We shall also, in some cases, find the lengths of the boundaries.

69. *To find expressions for the area and circumference of a circle in terms of its radius.*

From Lemma II. Cor. 2, it will be manifest that a circle is the limit of the inscribed regular polygon when the number of sides of the polygon is indefinitely increased.



Let ABC be a circle, radius r, and AC one side of an inscribed regular polygon of n sides;

$$\text{then the } \angle AOC = \frac{2\pi}{n}.$$

Now, by Art. 11, the area of the inscribed polygon is

$$\frac{nr^2}{2} \sin \frac{2\pi}{n};$$

$$\therefore \text{ area of the circle} = \text{limit of } \frac{nr^2}{2} \sin \frac{2\pi}{n},$$

when n is indefinitely increased;

$$\therefore \text{ area of the circle} = \text{limit of } \frac{2\pi}{n} \frac{nr^2}{2} \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}$$

$$= \text{limit of } \pi r^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}};$$

$$\text{but the limit of } \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1; \quad \text{Art. 66.}$$

$$\therefore \text{ area of the circle} = \pi r^2.$$

We may remark that πr^2 is the *superior* limit of the expression $\pi r^2 \dfrac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}$, since the ratio $\dfrac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}$ is always *less* than unity so long as n is finite, and only becomes equal to unity in the limit. This corresponds to the geometrical proposition, that the circle is the *superior* limit of the inscribed regular polygon.

We may obtain the same expression for the area of a circle by considering it to be the *inferior* limit of the *circumscribed* regular polygon.

Again, the circumference of a circle is the limit of the perimeter of the inscribed regular polygon.

Now the perimeter of this polygon $= 2nr \sin \frac{\pi}{n}$, Art. 11, Cor.;

$$\therefore \text{ circumference of the circle} = \text{limit of } 2nr \sin \frac{\pi}{n},$$

when n is indefinitely increased;

$$\therefore \text{ circumference of the circle} = \text{limit of } 2\pi r \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}};$$

$$\text{but the limit of } \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} = 1;$$

$$\therefore \text{ circumference of the circle} = 2\pi r.$$

70. The student must be cautioned against considering the latter part of the preceding Article as any *proof* that the circumference of a circle $= 2\pi r$. It amounts, at most, to a *verification* of a result previously ascertained; for, in all treatises on Trigonometry, π is taken to represent the ratio $\frac{\text{circumference}}{\text{diameter}}$, and therefore circumference $= 2\pi r$; but the *value* of π remains to be determined.

We have already shown that all circles are similar figures, and therefore, by Lemma V, their circumferences are proportional to their radii;

$$\therefore \text{ the ratio } \frac{\text{circumference}}{\text{radius}} \text{ is a } \textit{constant} \text{ ratio.}$$

If we denote this by 2π, then, since the circular measure of an angle is the ratio $\frac{\text{arc subtending the angle}}{\text{radius}}$, the circular measure of four right angles will be expressed by 2π, and the circular measure of half a right angle by $\frac{\pi}{4}$.

Now there is a theorem, called Gregory's Series, which expresses the circular measure of an angle in a series of powers of its tangent, which is as follows:—

$$\theta=\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\frac{\tan^7\theta}{7}+\&c.;$$

and this series will determine the numerical value of θ, provided θ be not greater than $\frac{\pi}{4}$. See Todhunter's *Trigonometry*, Art. 291, &c.

If $\theta=\frac{\pi}{4}$, $\tan\theta=1$;

$$\therefore \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\&c.$$

This is a numerical and convergent series, and the value of $\frac{\pi}{4}$ may be calculated from it. As, however, it does not converge with sufficient rapidity, another formula is employed which is as follows:—

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99};$$

and therefore, by the above theorem, we obtain

$$\begin{aligned}\frac{\pi}{4}=4&\left\{\frac{1}{5}-\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{5}\left(\frac{1}{5}\right)^5-\ldots\right\}\\ &-\left\{\frac{1}{70}-\frac{1}{3}\left(\frac{1}{70}\right)^3+\frac{1}{5}\left(\frac{1}{70}\right)^5-\ldots\right\}\\ &-\left\{\frac{1}{99}-\frac{1}{3}\left(\frac{1}{99}\right)^3+\frac{1}{5}\left(\frac{1}{99}\right)^5-\ldots\right\}.\end{aligned}$$

These series are rapidly convergent, and from them the value of π has been calculated to more than 400 places of decimals.

To thirteen places of decimals

$$\pi=3\cdot1415926535898.$$

Hence an approximate value of π is $\frac{22}{7}$; but a *much nearer* value is $\frac{355}{113}$, which is true as far as 6 places of decimals. The last fraction is easily remembered by writing down the first three odd numbers twice and dividing them in the middle: thus, 113 | 355.

Either of these fractions may be employed according to the degree of approximation required.

71. *To find the areas of a sector and segment of a circle.*

In the figure of Article 69, $AOBC$ is a sector, and ABC a segment of the circle.

Now by Euclid, VI. 33,

area of sector : area of circle :: arc AB : circumference,
:: angle AOB : 4 right angles.

Let A be the area of the sector, a the number of degrees in the angle AOB, and θ its circular measure;

$$\therefore A : \pi r^2 :: a^\circ : 360^\circ,$$

and

$$A : \pi r^2 :: \theta : 2\pi.$$

Hence, when the angle is expressed in degrees,

$$A = \pi r^2 \frac{a}{360};$$

and when the angle is expressed in circular measure,

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{r^2\theta}{2} = \frac{r\theta \times r}{2},$$

or

$$A = \frac{\text{arc} \times \text{radius}}{2}.$$

Again, let A' be the area of the segment ABC,

then A' = area of sector AOB − area of triangle AOB;

$$\therefore\ A'=\pi r^2\,\frac{a}{360}-\frac{r^2}{2}\sin a^\circ,$$

if the angle be expressed in degrees;

and
$$A'=\frac{r^2\theta}{2}-\frac{r^2\sin\theta}{2}$$
$$=\frac{r^2}{2}(\theta-\sin\theta),$$

if the angle be expressed in circular measure.

It will frequently happen that the radius and the angle AOB are not known, and cannot be measured; but they can always be found when we know the lengths of the chord AB and the versed sine CD.

For, let $AB=2a$, $CD=h$,

$$\text{then, since } OD^2=OA^2-AD^2,$$
$$\therefore\ (r-h)^2=r^2-a^2;$$
$$\therefore\ 2rh=a^2+h^2;$$
$$\therefore\ r=\frac{a^2+h^2}{2h}.$$

Also the angle $AOB=2$ angle BOC

$$=4 \text{ angle } BAC$$
$$=4\tan^{-1}\frac{h}{a}.$$

Thus the radius and the angle are known in terms of the chord and the versed sine.

We shall now obtain a formula, easy of application, when the length of the arc ACB can be measured, as well as the chord and versed sine.

Let the arc $ACB=2c$; and $OA=r$, $AB=2a$, $CD=h$, $AOB=\theta$, and the area $=A'$, as before.

Then $$A'=\frac{r^2}{2}(\theta-\sin\theta).$$

Now $$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$\text{but}\ \sin\frac{\theta}{2}=\frac{a}{r},\ \text{and}\ \cos\frac{\theta}{2}=\frac{r-h}{r};$$

$$\therefore\ \sin\theta=2\frac{a}{r}\,\frac{r-h}{r};\ \text{also}\ \theta=\frac{2c}{r};$$

$$\therefore\ A'=\frac{r^2}{2}\left\{\frac{2c}{r}-\frac{2a}{r^2}(r-h)\right\}$$

$$=cr-ar+ah$$

$$=(c-a)\,r+ah.$$

Again, $CD.(CO+OD)=AD^2$, Euclid, III. 35;

$$\therefore\ h\,(2r-h)=a^2;$$

$$\therefore\ r=\frac{a^2+h^2}{2h}.$$

Substituting this value of r in the last expression for A', we obtain

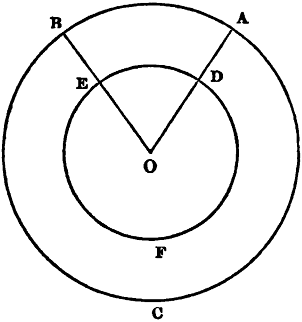
$$A'=\frac{(c-a)\,(a^2+h^2)}{2h}+ah$$

$$=\frac{ca^2+ch^2-a^3-ah^2+2ah^2}{2h};$$

$$\therefore\ A'=\frac{a^2(c-a)+h^2(c+a)}{2h}.$$

This formula gives the value of A' in terms of a, c and h, and is independent of the radius, and of the angle AOB subtended by the arc at the centre of the circle.

72. ***To find the area of an annulus,—or the space included between two concentric circles of unequal radii.***



Let R = radius of the outer circle ABC,

r = radius of the inner circle DEF,

and A = area of the annulus.

Then, manifestly,

$$\begin{aligned} A &= \pi R^2 - \pi r^2 \\ &= \pi(R^2 - r^2) \\ &= \pi(R+r)(R-r). \end{aligned}$$

We may also find a simple expression for the sectorial area $ABED$, in terms of the bounding arcs and the distance between them.

For, let C be the length of the arc AB,

c .... DE,

h the distance between them = AD.

Then area $ABED$ = sector AOB − sector DOE

$$= \frac{RC}{2} - \frac{rc}{2}; \quad \text{Art. } 71,$$

$$\therefore \text{ area of } ABED = \frac{(r+h)\,C - rc}{2}$$

$$= \frac{Ch + r\,(C-c)}{2}.$$

Now, since the arcs C and c subtend the same angle at O,

$$\frac{C}{c} = \frac{R}{r} = \frac{r+h}{r};$$

$$\therefore\ Cr = cr + ch,$$

$$\therefore\ r\,(C-c) = ch.$$

Substituting this value of $r\,(C-c)$ in the last expression for the area, we find

$$\text{area of } ABED = \frac{(C+c)\,h}{2}.$$

This formula is analogous to that of the trapezoid, C and c taking the place of the parallel sides.

We may remark that the area of the annular figure will be the same whether the circles are concentric or not, provided one circle is entirely included *within* the other.

73. *To find the area of a lune.*

A Lune is the figure formed by two segments of circles which have a common chord, and are on the same side of it.

If two circles be drawn intersecting one another, it will be seen that there are two lunes, one on each side of the common chord, and that the difference of the two lunes will always be equal to the annulus formed by the same circles.

To find the area of a lune, it will be necessary to find the area of each segment and to take their difference.

There is, however, one case in which the expression for the area of a lune takes a more simple form, and that is when the two circles have equal radii. In this case the centres of the two circles will always be on *opposite* sides of the common chord and equidistant from it; and if the area of one segment be

$$\frac{r^2}{2}(\theta-\sin\theta),$$

that of the other will be

$$\frac{r^2}{2}\{2\pi-\theta-\sin(2\pi-\theta)\},$$

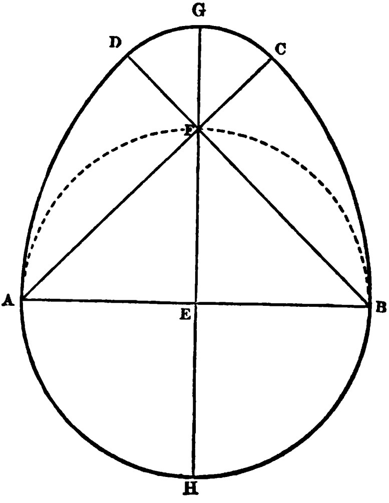
or

$$\frac{r^2}{2}(2\pi-\theta+\sin\theta).$$

The difference of these is $r^2(\pi-\theta+\sin\theta)$, which is therefore the area of the lune.

74. *To find the circumference and area of an oval formed by circular arcs.*

The simplest form of an oval is constructed as follows:—



Let ABF be a circle, AB a diameter, and E its centre. Draw EF perpendicular to AB. Join AF, BF, and produce them indefinitely.

With centres A, B describe the circular arcs BC, AD, and with centre F describe the arc CD. The curve $ABCD$ will be *one* form of an oval, and the simplest form.

It will be a *continuous* curve, because the tangents at C to the circular arcs CB, CD are both perpendicular to the straight line CA, and therefore coincide; and, for the same reason, the curve is continuous through the point B.

(1). *To find the circumference of the oval.*

Let r be the radius of the circle,

then the length of the semicircle $AHB = \pi r$;

and since the angle BAC is half a right angle, its circular measure $= \dfrac{\pi}{4}$;

$$\therefore \text{ arc } BC = 2r\,\frac{\pi}{4} = \frac{\pi r}{2},$$

and $\therefore$ the two arcs BC and AD together $= \pi r$.

Also the arc $CD = \dfrac{\pi}{2}\,CF$, since CFD is a right angle;

$$\therefore\ CD = \frac{\pi}{2}(AC - AF)$$

$$= \frac{\pi}{2}(2r - \sqrt{2}\,r)$$

$$= \pi r - \frac{\pi}{\sqrt{2}}\,r.$$

Hence the circumference of the oval $= 3\pi r - \dfrac{\pi}{\sqrt{2}}\,r$;

$$\therefore \text{ circumference} = \pi r\left(3 - \frac{1}{\sqrt{2}}\right).$$

(2). *To find the area of the oval.*

The area will manifestly be equal to the areas of the semicircle *AHB*, the quadrant *CDF*, and twice the area of the sector *BAC*, *minus* that of the triangle *ABF*.

If, then, *A* represent the area,

$$A = \frac{\pi}{2} EB^2 + \frac{\pi}{4} CF^2 + \frac{\pi}{4} AB^2 - EB^2$$

$$= \frac{\pi}{2} r^2 + \frac{\pi}{4} (2r - \sqrt{2}\, r)^2 + \frac{\pi}{4} (2r)^2 - r^2$$

$$= \frac{\pi}{2} r^2 + \pi r^2 (1 - \sqrt{2} + \tfrac{1}{2}) + \pi r^2 - r^2$$

$$= 3\pi r^2 - \pi \sqrt{2}\, r^2 - r^2 ;$$

$$\therefore\ A = \{\pi (3 - \sqrt{2}) - 1\}\, r^2.$$

Hence, also, the area of the luniform figure

$$AFBG = \{\pi (2 - \sqrt{2}) - 1\}\, r^2.$$

Various other forms of ovals may be constructed in a similar manner, by describing a number of arcs of circles so as to form a continuous curve symmetrical about the line *GH*, which is called the axis of the oval; and their areas may in all cases be found by taking the difference between a series of circular arcs and a series of triangles.

These curves are of great importance. They are the forms given to the transverse sections of railway tunnels, large drains, and many other works.

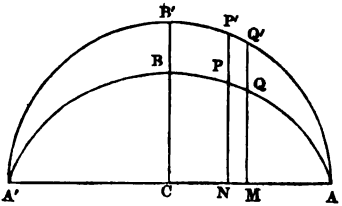
75. *To find the area of an ellipse.*

Let *AA′* be the axis-major of an ellipse, *BC* the semi-axis-minor, *AB′A′* the semicircle on the axis-major as diameter.

Through any two points *N*, *M*, near to each other, draw the ordinates *NPP′*, *MQQ′* parallel to *BC*, and draw the chords *PQ* and *P′Q′*.

Then the two rectilinear figures PM, $P'M$ are trapezoids; and if A, A' represent their areas, we have

$$\frac{A}{A'}=\frac{NP+MQ}{NP'+MQ'}.$$



This will be true however small NM may be, but, ultimately, when NM is diminished indefinitely,

$$MQ=NP, \text{ and } MQ'=NP';$$

$$\therefore, \text{ ultimately, } \frac{A}{A'}=\frac{2NP}{2NP'}=\frac{NP}{NP'}.$$

Now it is a property of the ellipse that

$$\frac{NP}{NP'}=\frac{CB}{CB'}=\frac{BC}{AC};$$

$$\therefore, \text{ ultimately, } \frac{A}{A'}=\frac{BC}{AC}.$$

Hence, in the semi-ellipse and in the semicircle, there can be inscribed the same number of rectilinear figures which are ultimately to each other in the ratio of BC to AC; therefore, by Lemma IV, the semi-ellipse and the semicircle, and therefore also the ellipse and the circle are to each other in this same ratio;

$$\therefore \frac{\text{area of ellipse}}{\text{area of circle}}=\frac{BC}{AC}.$$

If $AC=a$, $BC=b$, the area of the circle will be πa^2;

$$\therefore \text{ area of ellipse} = \pi a^2 \frac{b}{a} = \pi ab.$$

Cor. 1. The area of any segment of the ellipse, cut off by a line parallel to the minor axis, will be to the corresponding segment of the circle upon the major axis in the ratio of $b : a$; for there can be inscribed in the two segments the same number of rectilinear figures which are ultimately to each other in this ratio.

Cor. 2. In the same manner we may show that the area of the ellipse will be to the area of the circle on the minor axis in the ratio of $a : b$; and that any segment of the ellipse, cut off by a line parallel to the major axis, will be to the corresponding segment of the circle on the minor axis in the ratio of $a : b$.

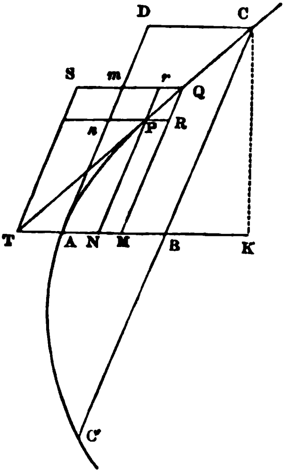
In the proof we have used the *trapezoids* $PNMQ$, $P'NMQ'$, but the same result may be obtained from the *parallelograms* inscribed in the figures $PNMQ$, $P'NMQ'$, since these parallelograms are ultimately to each other in the ratio $BC : AC$, and their sums are ultimately the areas of the semi-ellipse and the semicircle.

76. *To find the area of a segment of a parabola cut off by any straight line.*

Let CAC' be a segment of a parabola cut off by the straight line CC'. Bisect CC' in B, and draw BA parallel to the axis of the parabola. Through A draw the tangent AD, which will be parallel to BC, and complete the parallelogram $ABCD$. We shall show that the parabolic area ABC is equal to two-thirds of the parallelogram $ABCD$.

In the curve AC take any two points P, Q, near to each other; join QP, and produce it to meet BA produced in T. Through P, Q draw lines parallel to BC and AB, and complete the parallelogram MS, as shown in the figure.

Now since PM and PS are complements about the diameter of the parallelogram MS, they are equal to each other, always, and therefore ultimately; but, ultimately, when Q moves up to P, PT becomes the tangent at P, and $AT = AN$ (by a property of the parabola): therefore, ultimately, the parallelogram PS is double of the parallelogram Pm, and therefore also the parallelogram PM is ultimately double of the parallelogram Pm.



Hence, in the two curvilinear figures ACB, ACD, there can be inscribed the same number of parallelograms which are ultimately to each other in the ratio 2 : 1; therefore, by Lemma IV, the curvilinear areas are to each other in this ratio,

i.e. area ACB is double of the area ACD;

and $\therefore$ area ACB is $\frac{2}{3}$ of the parallelogram $ABCD$.

Similarly, area $AC'B$ is $\frac{2}{3}$ of the parallelogram ABC'.

Hence the parabolic segment CAC' is $\frac{2}{3}$ of the circumscribing parallelogram.

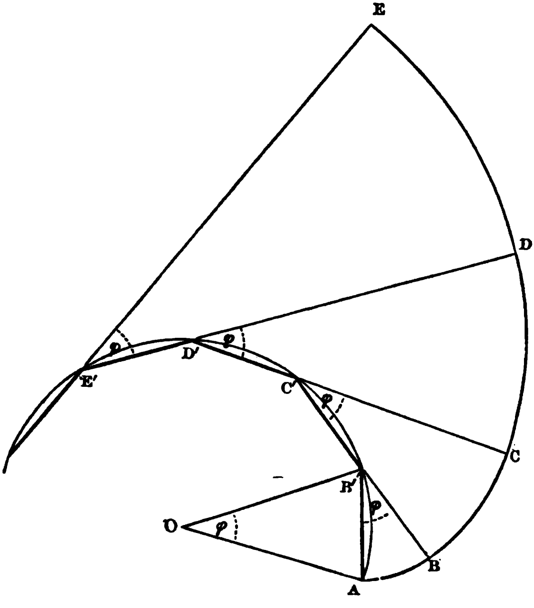
Cor. Hence

$$\text{area } CAC' = \frac{2}{3} AB \times CC' \times \sin ABC$$
$$= \frac{2}{3} AB \times 2BC \times \sin ABC$$
$$= \frac{2}{3} AB \times 2CK, \text{—if } CK \text{ be perpendicular to } AB.$$

This is the property which we employed in Art. 26.

Def. A Spiral is the name given to a curve traced by a point which revolves round a fixed point, and always increases, or always decreases its distance from that point, according to a given law.

77. *To find the length and area of the spiral called the involute of a regular polygon.*



Let $AB'C'D'E'$ be a portion of a regular polygon, and suppose the extremity of a fine inextensible string coiled round the polygon to reach to the point A. If the string be uncoiled, and always kept stretched, the extremity of it will manifestly trace a curve of the form $ABCDE$, &c., consisting of a series of *circular arcs* AB, BC, CD, DE, &c., the radii of which will be as the numbers 1, 2, 3, 4, &c., that of the least being the length of a side of the polygon.

The curve $ABCDE$, &c., is called the *involute* of the polygon, and the polygon the *evolute* of this curve.

It will be a *continuous* curve, since any two consecutive arcs, as BC, CD, have a common tangent at C, viz, the straight line which is perpendicular to CD'.

(1). *To find the length of the involute after the string has been uncoiled from* p *consecutive sides of the polygon.*

Let n be the number of sides of the polygon, a the length of one side, ϕ the exterior angle of the polygon, and therefore equal to the angle subtended at the centre by any one side.

Then $$AB = a\phi,\ BC = 2a\phi,\ CD = 3a\phi,\ \&c.,$$

$$\therefore AB + BC + CD + \&c. = a\phi + 2a\phi + 3a\phi + \&c.;$$

$$\therefore \text{length of the involute} = a\phi\,(1 + 2 + 3 + \ldots + p)$$

$$= a\phi\frac{p\,(p+1)}{2} = \frac{\pi a}{n}\,p\,(p+1), \text{ since } \phi = \frac{2\pi}{n}.$$

If the string make one complete revolution, $p = n$, and the length of the involute after one revolution of the string, or the length of the *first convolution* $= \pi a\,(n+1)$.

Again, if the string make r revolutions, $p = rn$,

$\therefore$ length of the *first* r convolutions $= \pi ar\,(rn+1)$,

and, similarly, the length of the *first* $(r+1)$ convolutions

$$= \pi a\,(r+1)\,\{(r+1)\,n+1\};$$

and the difference of these expressions will manifestly be the length of the $(r+1)^{\text{th}}$ convolution;

$\therefore$ length of the $(r+1)^{\text{th}}$ convolution

$$=\pi a\{(r+1)^2 n+r+1-r^2 n-r\}$$

$$=\pi a(2rn+n+1).$$

From this general expression the length of any convolution may be found by making $(r+1)$ equal to the number of the convolution.

Thus, length of *first* convolution $=\pi a(n+1)$,

.... *second* .... $=\pi a(3n+1)$,

.... *third* .... $=\pi a(5n+1)$,

.... $=$ ....,

.... $(r+1)^{\text{th}}$ .... $=\pi a\{(2r+1)n+1\}$.

The sum of these $(r+1)$ convolutions

$$=\pi a\{n(1+3+5+\ldots+\overline{2r+1})+r+1\}$$

$$=\pi a\{(r+1)^2 n+r+1\}$$

$$=\pi a(r+1)\{(r+1)n+1\},$$

which is identical with the expression above for the length of the *first* $(r+1)$ convolutions.

(2). *To find the area included between the involute, the polygon, and the last position of the string.*

This area will manifestly be the sum of the circular sectors $AB'B$, $BC'C$, $CD'D$, &c.;

$$\therefore \text{Area}=\frac{a^2\phi}{2}+\frac{(2a)^2\phi}{2}+\frac{(3a)^2\phi}{2}+\ldots\ldots+\frac{(pa)^2\phi}{2},$$

$$=\frac{a^2\phi}{2}\{1^2+2^2+3^2+\ldots\ldots\ldots+p^2\}$$

$$=\frac{a^2\phi}{12}p(p+1)(2p+1), \quad \text{Art. 67,}$$

or, since $\phi = \frac{2\pi}{n}$,

$$\therefore \text{ Area } = \frac{\pi a^2}{6n} p(p+1)(2p+1).$$

If the string make one complete revolution, $p = n$;

$\therefore$ area swept over by the string in its *first* revolution

$$= \frac{\pi a^2}{6}(n+1)(2n+1).$$

Similarly, making $p = rn$, the area swept over by the string in the *first* r revolutions

$$= \frac{\pi a^2}{6} r(rn+1)(2rn+1),$$

and the area swept over in the *first* $(r+1)$ revolutions

$$= \frac{\pi a^2}{6}(r+1)\{(r+1)n+1\}\{2(r+1)n+1\}.$$

The difference of these expressions is manifestly the area swept over in the $(r+1)^{\text{th}}$ revolution of the string, *and therefore the area of the curve described.* If we denote this area by A_{r+1},

$$\begin{aligned} A_{r+1} &= \frac{\pi a^2}{6}\{(r+1)(\overline{r+1}\,n+1)(2\,\overline{r+1}\,n+1) \\ &\qquad -r(rn+1)(2rn+1)\} \\ &= \frac{\pi a^2}{6}\{(r+1)(2\,\overline{r+1}^2 n^2+3\,\overline{r+1}\,n+1) \\ &\qquad -r(2r^2n^2+3rn+1)\} \\ &= \frac{\pi a^2}{6}\{(r+1)(2r^2n^2+4rn^2+2n^2+3rn+3n+1) \\ &\qquad -r(2r^2n^2+3rn+1)\} \\ &= \frac{\pi a^2}{6}(6r^2n^2+6rn^2+6rn+2n^2+3n+1); \end{aligned}$$

$$\therefore A_{r+1} = \frac{\pi a^2}{6}\{6rn(rn+n+1)+2n^2+3n+1\}.$$

Similarly, if A_r denote the area of the *first* r convolutions,

$$A_r = \frac{\pi a^2}{6} \{6(r-1)n(\overline{r-1}\,n+n+1)+2n^2+3n+1\};$$

and the difference,

$$A_{r+1}-A_r,$$

will be the area of the $(r+1)^{\text{th}}$ convolution.

By subtraction, we find

$$A_{r+1}-A_r = \pi a^2 n(2rn+1).$$

From this expression we may find the areas of the successive convolutions described, *after the first revolution of the string*, by giving to r the values 1, 2, 3, &c.

If we give to r all the values 1, 2, 3, &c. up to and including r, and add together the resulting quantities, and also the area described in the *first* revolution of the string, their sum will be found identical with the expression for A_{r+1}, as it manifestly ought to be.

78. From the expressions of the preceding Article for the length and area of the involute of a regular polygon, we may find the length and area of the spiral called the Involute of the Circle, by supposing the number of sides of the polygon to be indefinitely increased.

For this purpose we must substitute for a its value in terms of n and the radius of the *inscribed* circle, and for p its value in terms of n and the angle subtended at the centre by p sides of the polygon; and then take the limits of these expressions when n is indefinitely increased.

If r be the radius of the inscribed circle, and θ the angle subtended at the centre by p sides of the polygon, these values will be

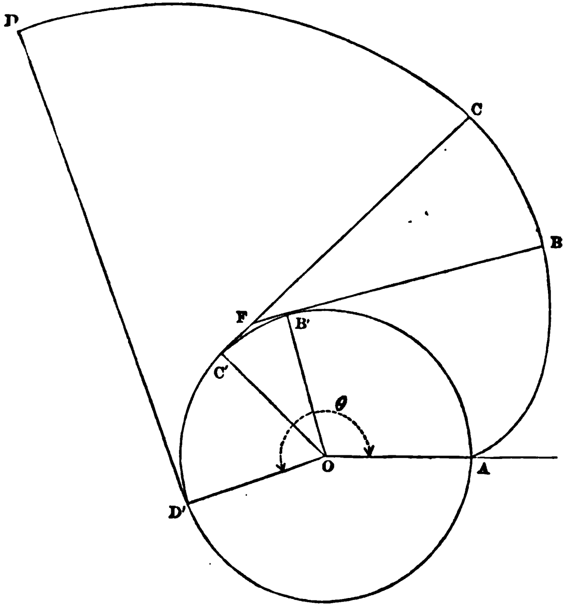
$$a = 2r\tan\frac{\pi}{n}, \quad \text{and } p = \frac{n}{2\pi}\,\theta.$$

We shall, however, give an independent and simpler investigation for the Involute of the Circle in the next Article.

It will be manifest that every polygon will have an involute, which will be a continuous curve consisting of a series of circular arcs, whether the polygon have or have not re-entering angles. Thus, half the oval in Art. 74, will evidently be the involute of the portion *HEAFG* of an irregular polygon, having a re-entering angle at *A*.

The length of the involute of any polygon may be found by taking the sum of the circular arcs of which it is composed; and the area will be the sum of the circular sectors swept over by the string.

79. *To find the area and length of the involute of the circle.*



Def. If a fine inextensible string, coiled round a circle, be uncoiled from it, and the string be kept always stretched, its extremity will trace the spiral called the Involute of the Circle.

Let $ABCD$ be a portion of the involute of the circle $AB'C'D'$, the extremity of the string when coiled upon the circle being at A.

We shall find an expression for the area included between the involute, the circle, and DD' the last position of the string.

Let B, C be two points in the involute near to each other. Draw BB', CC' tangents to the circle intersecting in F, and join OB', OC'; then the angle BFC is equal to the angle $B'OC'$; and from the nature of the curve,

$$BB' = \text{arc } AB', \quad CC' = \text{arc } AC'.$$

Let a be the radius of the circle, and θ the circular measure of the angle AOD'; and suppose this angle to be divided into n equal parts, of which $B'OC'$ is one, and AOB' the first r of these parts measured from OA, so that

$$B'OC' = \frac{\theta}{n}, \quad AOB' = r\frac{\theta}{n}.$$

Now the sector BFC is ultimately a sector of a circle, and its area is, ultimately,

$$\frac{1}{2} BF^2 \times \text{circular measure of } BFC;$$

$$\text{and} \quad BFC = B'OC' = \frac{\theta}{n},$$

$$\therefore \text{, ultimately, area of sector } BFC = \frac{1}{2} BF^2 \frac{\theta}{n};$$

but, ultimately, when BC is indefinitely diminished,

$$BF = BB' = AB' = AO \times \text{c.m. of } AOB' = ar\frac{\theta}{n};$$

∴, ultimately,

$$\text{area of sector } BFC = \frac{1}{2} a^2 r^2 \frac{\theta^3}{n^3}.$$

Hence the sum of the areas of all the sectors which ultimately form the curvilinear area

$$=\frac{1}{2}a^2\frac{\theta^3}{n^3}\{1^2+2^2+3^2+\ldots\ldots+(n-1)^2\}.$$

Hence the area of the curvilinear figure bounded by the involute, the circle, and the straight line DD', is the limit of this expression when n is indefinitely increased;

$$\therefore \text{ curvilinear area } =\frac{a^2\theta^3}{6}. \quad \text{Art. 67, Cor.}$$

If we represent by R the length of the string unwound,

$$\therefore \quad R=a\theta;$$

$$\therefore \text{ curvilinear area}=\frac{a^3\theta^3}{6a}=\frac{R^3}{6a}.$$

This formula is readily applied, because the length R can easily be measured.

If the string make one complete revolution, $\theta=2\pi$, and $R=2\pi a$;

$\therefore$ area swept over by the string in the *first* revolution

$$=\frac{2^3\pi^3a^2}{6}=\frac{2^2\pi^3a^2}{3}.$$

Similarly, the area swept over by the string in the *first* r revolutions

$$=\frac{2^2r^3\pi^3a^2}{3};$$

and the area swept over in the *first* $(r+1)$ revolutions

$$=\frac{2^2(r+1)^3\pi^3a^2}{3};$$

and the difference of these will be the area swept over by the string in the $(r+1)^{\text{th}}$ revolution, *and therefore the area of the curve described.* Let A_{r+1} denote this area,

$$\therefore A_{r+1} = \frac{2^2\pi^3a^2}{3}\{(r+1)^3 - r^3\}.$$

Similarly, if A_r denote the area of the *first* r convolutions,

$$A_r = \frac{2^2\pi^3a^2}{3}\{r^3 - (r-1)^3\};$$

and the difference, $A_{r+1} - A_r$, will be the area of the $(r+1)^{\text{th}}$ convolution. Taking this difference, we find

$$A_{r+1} - A_r = \frac{2^2\pi^3a^2}{3}\{(r+1)^3 - 2r^3 + (r-1)^3\},$$

or

$$A_{r+1} - A_r = \frac{2^2\pi^3a^2}{3}\,6r;$$

$\therefore$ area of $(r+1)^{\text{th}}$ convolution $= 6r$ times the area of the *first* convolution.

From this equation the area of any one convolution may be found; and the expression for A_{r+1} will give the area of the whole curve.

In the preceding investigation we have supposed the string to make an *exact* number of revolutions. If we now suppose it to describe an angle θ in addition to r complete revolutions, the whole angle described will be $2r\pi + \theta$. Let ϕ denote this angle, and let R_ϕ denote the length of the string unwound. Then the whole area swept over

$$= \frac{R_\phi{}^3}{6a}.$$

Now if the string make one more revolution, ϕ will become $2\pi + \phi$, which we may represent by ϕ_1, and the whole area swept over will be

$$\frac{R_{\phi_1}^3}{6a}.$$

Hence the difference,

$$\frac{1}{6a}\{R_{\phi_1}^3 - R_\phi{}^3\},$$

is the area of the curve described when the string has revolved through an angle

$$2(r+1)\pi+\theta.$$

The lengths R_ϕ and R_{ϕ_1} are easily measured, and their difference will evidently be the circumference of the circle, or $2\pi a$.

(2). *To find the length of any portion of the involute.*

The extremity of the string which traces the curve is at the point B, *moving in a circle* of which B' is the centre; therefore, ultimately, BC is the arc of a circle the radius of which is BB', and therefore $\frac{BC}{BF}$, or $\frac{BC}{BB'}$ is ultimately the circular measure of the angle BFC, i. e. of the angle $B'OC'$,

$$\text{therefore, ultimately, } \frac{BC}{BB'}=\frac{\theta}{n},$$

$$\text{but } BB'=\text{arc } AB'=ar\frac{\theta}{n};$$

$$\text{therefore, ultimately, } BC=ar\frac{\theta^2}{n^2}.$$

Hence, the length of the involute from A to D is the limit of $\frac{a\theta^2}{n^2}(1+2+3+\ldots+n)$, when n is indefinitely increased;

$\therefore$ length of the involute $=\frac{a\theta^2}{2}$, or $=\frac{R^2}{2a}$, if R be the length of the string unwound.

If the string make one complete revolution, $\theta=2\pi$, and $R=2\pi a$;

$$\therefore \text{ length of } \textit{first} \text{ convolution} = 2\pi^2 a.$$

Similarly, the length of the *first* r convolutions

$$=2\pi^2 r^2 a,$$

and the length of the *first* $(r+1)$ convolutions

$$= 2\pi^2 (r+1)^2 a;$$

$\therefore$ the length of the $(r+1)^{\text{th}}$ convolution

$$= 2\pi^2 \{(r+1)^2 - r^2\} a$$

$$= 2\pi^2 (2r+1) a.$$

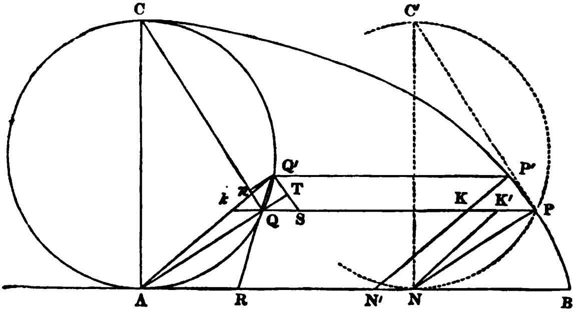
From this expression the length of any one convolution may be found by making $(r+1)$ equal to the number of the convolution.

Also, if the string describe an angle θ in addition to an exact number of revolutions, and if ϕ represent the whole angle described, and R_ϕ the length of the string unwound,

$$\therefore \text{ length of involute} = \frac{R_\phi^2}{2a}.$$

80. *To find the length and area of a cycloid.*

Def. If a circle *roll* upon a fixed straight line, any point in its circumference will trace a curve called a Cycloid.



Let AQC be the generating circle, C the generating point when in its highest position.

If the circle roll upon the line AB, the point C will manifestly trace a curve of the form represented by CPB. AB will be half the circumference of the generating circle, and BPC will be half the cycloid, the other half being symmetrically situated on the other side of the line AC. This line is called the axis of the cycloid, the point C the vertex, and AB the base.

Let NPC' be the position of the generating circle when the point C has moved to P.

Now, from the mode of generating the curve, it will be manifest that the generating point, when at P, is *moving in a circle* about some point in PN produced; therefore the tangent at P will be at right angles to PN, and therefore in the direction of the chord PC'.

(1). *To find the length of the arc* CP *measured from the vertex.*

Let P' be a point in the cycloid near to P.

Draw PQ, $P'Q'$ parallel to AB. Join AQ, CQ, and draw $Q'S$ parallel to CQ, intersecting AQ produced, in T; join $Q'Q$, and produce it to meet AB in R, and with centre C describe the circular arc $Q'n$.

Now PC', the tangent at P, is parallel to CQ, and, ultimately, when P' moves up to P, and therefore Q' to Q,

$$PP' = Q'S.$$

But, ultimately, when Q' moves up to Q, RQ' becomes the tangent to the circle at Q, and the $\angle RQA = \angle RAQ$; but the $\angle RQA = \angle Q'QT$, and the $\angle RAQ = \angle SQT$,

$$\text{therefore, ultimately, } \angle Q'QT = \angle SQT;$$

also the angles QTS, QTQ' are equal, each being a right angle.

Therefore the triangles QTS, QTQ' are ultimately equiangular, and, since they have the side QT common, they are ultimately equal in all respects.

$$\text{Therefore, ultimately, } Q'T = TS, \text{ or } Q'S = 2Q'T;$$

$$\text{but, ultimately, } Q'T = Qn = CQ - CQ', \text{ and } PP' = Q'S,$$

$$\therefore, \text{ ultimately, } PP' = 2\,(CQ - CQ');$$

in other words, the arc CP increases *twice as fast* as the chord CQ; and these lines commence together at the point C; therefore the arc CP will be double of the chord CQ.

Hence the length of the semicycloid $CB = 2CA$, and therefore the length of the cycloid is equal to four times the diameter of the generating circle.

(2). *To find the area of a cycloid.*

In the figure join AQ', and draw $P'N'$ parallel to it, intersecting PQ in K; produce PQ to meet AQ' in k, and draw NK' parallel to $N'P'$; then the triangles PNK' QAk will be equal in all respects.

Again, the triangles KPP', kSQ' are ultimately similar and equal to each other;

$$\therefore\ KP \text{ is ultimately equal to } kS;$$

but kQ' is ultimately parallel to QT, and $Q'S$ is ultimately double of $Q'T$, and therefore kS is ultimately double of kQ.

Hence KP is ultimately double of kQ, and therefore of $K'P$;

$$\therefore\ KK' \text{ is ultimately equal to } K'P;$$

$\therefore$ the parallelogram KN is ultimately double of the triangle PNK', and therefore of the triangle QAk.

Hence the trapezoid $KPNN'$ is ultimately three times the triangle QAk.

Hence, in the semicycloid and in the semicircle there can be inscribed the same number of rectilinear figures which are ultimately to each other in the ratio 3 : 1; and their sums are ultimately the areas of the semicycloid and the semicircle. Therefore, by Lemma IV. Cor.,

$$\frac{\text{area of semicycloid}}{\text{area of semicircle}} = \frac{3}{1};$$

$$\therefore \text{area of cycloid} = 3 \text{ area of generating circle.}$$

We may now find the centre of the circle which the generating point is describing when at the point P; for, since $KK'=NN'$, PK is ultimately double of NN', and therefore, when PP' is indefinitely diminished, PN, $P'N'$ will intersect in a point in PN produced, the distance of which from P is $2PN$.

It is round this point that the generating point is *moving in a circle* when at P.

This circle is called the "circle of curvature" at the point P, being the circle which coincides with the curve at that point more nearly than any other circle; and its radius is called the "radius of curvature" at that point.

Since P is *any* point in the cycloid, it will follow that all the radii of curvature are bisected in the base AB; therefore the radius of curvature at the point C is $2CA$, and at the point B, zero. Hence it may easily be proved that the centres of the circles of curvature all lie in another semicycloid which has its vertex at B, and its axis parallel to AC; and therefore every semicycloid is the *involute* of an equal semicycloid in this position.

CHAPTER VIII.

AREAS AND VOLUMES OF SOLIDS BOUNDED BY CURVE SURFACES.

81. *To find the area of the surface, and the volume, of a circular cylinder.*

Def. A circular cylinder is the solid generated by the revolution of a rectangle about one of its sides which remains fixed: this line is called the axis of the cylinder. From Lemma II. Cor. 2, it will be manifest that a circular cylinder is the limit of a right prism, the base of which is a regular polygon, when the number of sides of the base is increased indefinitely, and the magnitude of each indefinitely diminished;—the radius of the circle circumscribing the polygon remaining unaltered, and being the radius of the base of the cylinder.

Now, by Art. 12, the area of the lateral surface of a right prism is equal to the perimeter of the base multiplied by the altitude, and this will be true always, whatever may be the number of sides of the base, and therefore true of the limit, i. e. of the cylinder,

$\therefore$ area of convex surface of cylinder

$$=(\text{circumference of base})\times\text{altitude}.$$

Let r be the radius of the base of the cylinder, and therefore $2\pi r$ its circumference; and let h be the altitude;

$$\therefore \text{ area of convex surface} = 2\pi r h,$$

$$\text{also area of each end} = \pi r^2;$$

$$\therefore \text{ area of whole surface of cylinder} = 2\pi r\,(r + h).$$

(2). *To find the volume of the cylinder.*

Since the volume of the right prism, of which the cylinder is the limit, is equal to the area of the base multiplied by the altitude, (Art. 38), by the same reasoning as before, this will be true of the cylinder;

$$\therefore \text{ volume of cylinder} = (\text{area of base}) \times \text{altitude}.$$

Now area of base $= \pi r^2$, and altitude $= h$,

$$\therefore \text{ volume of cylinder} = \pi r^2 h.$$

82. *To find the area of the surface, and the volume, of any cylinder.*

Every cylinder may be considered the limit of a prism, when the number of sides of the prism is increased indefinitely, and the breadth of each indefinitely diminished, —the perimeter of its transverse section remaining finite.

Now, by Art. 13, the area of the lateral surface of any prism is equal to the perimeter of its transverse section multiplied by the length; and, by Art. 38, the volume of any prism is equal to the area of the base multiplied by the altitude, or, to the area of the transverse section multiplied by the length; and this being true always, will be true of the limit, and therefore true of the cylinder.

$\therefore$ area of convex surface of cylinder

$$= (\text{perimeter of transverse section}) \times \text{length};$$

and volume of cylinder

$$= (\text{area of base}) \times \text{altitude},$$

or $$= (\text{area of transverse section}) \times \text{length}.$$

Let A be the area of the base, h the altitude, and l the length of the cylinder; p the perimeter and A' the area of the transverse section, S the area of the convex surface, and V the volume;

$$\therefore S=pl; \quad V=Ah, \quad \text{or} \quad V=A'l.$$

83. *To find the volume of any frustum of a circular cylinder.*

In Art. 42, we proved that the volume of any frustum of a regular prism is equal to the area of its transverse section multiplied by the length of the *axis*. This is true always, whatever may be the number of sides of the prism, and therefore true of the limit, i.e. of the frustum of a circular cylinder. Thus, if r be the radius of the transverse section, l the length of the axis,

$$\therefore \text{volume} = \pi r^2 l.$$

The length l can easily be found, for if h_1, h_2 be any two lengths of the solid, diametrically opposite to each other, then

$$2l = h_1 + h_2.$$

84. *To find the area of the surface, and the volume, of a circular cone.*

DEF. A circular cone is the solid generated by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed: this line is called the axis of the cone. From Lemma II. Cor. 2, it will be manifest that a circular cone is the limit of a regular pyramid when the number of sides of the base is increased indefinitely, and the magnitude of each indefinitely diminished;—the radius of the circle circumscribing the base remaining unaltered, and being the radius of the base of the cone.

Now, by Art. 15, the area of the sloping surface of a regular pyramid is equal to half the perimeter of the base multiplied by the slant height, and, by reference to the figure of Art. 15, it will be seen that the slant height of the pyramid is ultimately equal to the slant height of the cone.

Again, by Art. 39, the volume of a pyramid is equal to one-third of the area of the base multiplied by the altitude.

And whatever is true of the pyramid will be true of its limit, i.e. of the cone;

$\therefore$ area of convex surface of cone

$$=\frac{1}{2}\text{(circumference of base)}\times\text{slant height;}$$

and volume of cone

$$=\frac{1}{3}\text{(area of base)}\times\text{altitude.}$$

Let r be the radius of the base, s the slant height, and h the altitude of the cone;

$$\therefore\ \text{area of convex surface}=\frac{1}{2}2\pi rs=\pi rs,$$

$$\text{and volume of cone}=\frac{1}{3}\pi r^2h.$$

It is sometimes required to find the surface and volume of a cone, in terms of the altitude and the vertical angle.

Let a be the vertical angle of the cone;

$$\therefore\ s=h\sec\frac{a}{2},\quad\text{and } r=h\tan\frac{a}{2};$$

$$\therefore\ \text{area of convex surface}=\pi h^2\sec\frac{a}{2}\tan\frac{a}{2},$$

$$\text{and volume of cone}=\frac{1}{3}\pi h^3\tan^2\frac{a}{2}.$$

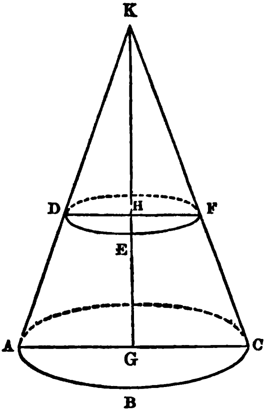
85. *To find the volume of any cone.*

Every cone may be considered the limit of a pyramid when the number of sides of the base is increased indefinitely, and the magnitude of each indefinitely diminished, —the perimeter of the base remaining finite.

Now, by Art. 39, the volume of *any* pyramid is equal to one-third of the area of the base multiplied by the altitude, and this being true always, will be true of the limit, i.e., will be true of the cone;

$$\therefore \text{ volume of any cone} = \frac{1}{3} \text{ (area of base)} \times \text{altitude.}$$

86. *To find the area of the convex surface of a frustum of a circular cone, made by a plane perpendicular to the axis.*



Let *ABC* be the base, and *K* the vertex of a circular cone; and let the frustum *ABC . DEF* be cut off by a plane *DEF* perpendicular to the axis, and therefore parallel to the base.

Let R = radius of base of frustum = AG,

r = radius of top of frustum = DH,

s = slant height of frustum = AD,

and l = slant height of cone cut off = DK.

Therefore area of convex surface of frustum

$$= \pi R(s+l) - \pi rl$$
$$= \pi\{(R-r)l + Rs\};$$

and from this expression l must be eliminated, because it is unknown.

Now, by similar triangles AGK, DHK,

$$\frac{R}{r} = \frac{s+l}{l}, \text{ and } \therefore \frac{R-r}{r} = \frac{s}{l};$$

$$\therefore (R-r)l = rs.$$

Substituting this value of $(R-r)l$ in the expression for the area, we have

$$\text{area of convex surface} = \pi(Rs + rs) = \pi s(R+r).$$

The expression for the area of the convex surface of a cone, viz. πrs, may be written

$$\frac{1}{2} . 2\pi r . s,$$

which shows that the surface of a cone is equal to the area of a sector of a circle, the radius of which is equal to the slant height of the cone, and the arc to the circumference of its base.

Also the expression,

$$\pi s(R+r) \text{ or } \frac{1}{2}(2\pi R + 2\pi r)s,$$

shows that the surface of a frustum of a cone is equal to the sector of an annulus, the bounding arcs of which are the circumferences of the base and top of the frustum, and the distance between them its slant height. See Art. 72.

87. *To find the volume of a frustum of a circular cone, made by a plane perpendicular to the axis.*

Making use of the figure of the preceding Article, let R, r be the radii of the base and top, as before,

h = altitude of frustum $= GH$,

k = altitude of cone cut off $= HK$,

and V = volume of frustum;

$$\therefore\ V = \frac{1}{3}\pi R^2(h+k) - \frac{1}{3}\pi r^2 k,$$

$$= \frac{\pi}{3}\{R^2 h + (R^2 - r^2)k\};$$

and from this expression for V, k must be eliminated because it is unknown.

Now, by similar triangles AGK, DHK,

$$\frac{R}{r} = \frac{h+k}{k},$$

$$\therefore\ \frac{R-r}{r} = \frac{h}{k}, \text{ and } \therefore\ \frac{R^2 - r^2}{Rr + r^2} = \frac{h}{k},$$

$$\therefore\ (R^2 - r^2)k = (Rr + r^2)h.$$

Substituting this value of $(R^2 - r^2)k$ in the expression for V, we have

$$V = \frac{\pi h}{3}\{R^2 + Rr + r^2\}.$$

88. *We may also find the volume of a frustum of any cone, made by a plane parallel to the base.*

For this solid will manifestly be the limit of a frustum of a pyramid, the expression for the volume of which is

$$V = \frac{h}{3}\{A_1 + \sqrt{(A_1 A_2)} + A_2\}, \text{ Art. 40,}$$

where h is the altitude of the frustum, and A_1, A_2 are the areas of the base and top; and this expression being true always will be true in the limit, and will therefore determine the volume of a frustum of *any* cone.

89. The formula obtained in Art. 87, for the frustum of a circular cone, will evidently be a particular case of the formula of Art. 40, and may be deduced from it by supposing the two *similar* polygons which form the base and top of the frustum to be *regular* polygons, and to become circles in the limit; and the straight line joining their centres to be perpendicular to their planes.

Let the radii of these circles be R and r,

then $$A_1 = \pi R^2, \qquad A_2 = \pi r^2,$$

and $$\sqrt{(A_1 A_2)} = \sqrt{(\pi^2 R^2 r^2)} = \pi R r.$$

Making these substitutions in the formula

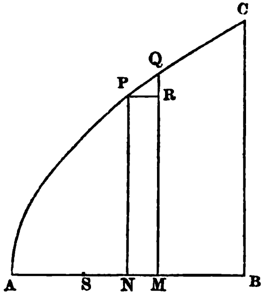
$$V = \frac{h}{3}\{A_1 + \sqrt{(A_1 A_2)} + A_2\},$$

we find

$$V = \frac{\pi h}{3}(R^2 + Rr + r^2),$$

which is identical with the expression in Art. 87.

90. *To find the volume of a segment of a paraboloid of revolution, made by a plane perpendicular to the axis.*



Let AC be an arc of a parabola, AB its axis, S its focus, and BC an ordinate perpendicular to the axis.

Divide AB into n equal parts, of which NM is one, and AN the first r of these parts measured from A. Through N, M draw NP, MQ parallel to BC, and draw PR parallel to AB.

Let $AS=a$, $AB=h$, $BC=k$;

$$\therefore\ AN=r\frac{h}{n},\quad NM=\frac{h}{n}.$$

If the whole figure revolve round AB fixed, the parabolic area ABC will generate a segment of a paraboloid of revolution; and the rectangle MP will generate a cylinder, the volume of which will be

$$\pi.NP^2.NM.$$

Now, by a well-known property of the parabola,

$$NP^2=4AS.AN=4ar\frac{h}{n};$$

$$\therefore \text{ volume of cylinder generated by } MP=4\pi ar\frac{h^2}{n^2}.$$

Hence the volume of the segment of the paraboloid of revolution will be the limit of the expression

$$4\pi ah^2\frac{1}{n^2}\{1+2+3+\ldots+(n-1)\},$$

when n is indefinitely increased;

$$\therefore \text{ volume of paraboloid}=\frac{1}{2}4\pi ah^2=2\pi ah^2.$$

This expression may be written

$$\frac{1}{2}\pi h\,4ah, \text{ or } \frac{1}{2}\pi k^2h, \text{ since } k^2=4ah;$$

$\therefore$ volume of paraboloid $=\frac{1}{2}$ volume of circumscribing cylinder.

91. *To find the volume of any frustum or zone of a paraboloid, contained between two planes perpendicular to the axis.*

Let R, r be the radii of the two parallel ends of the frustum, k its thickness, and V its volume. Also let h and h_1 be the altitudes of the two segments whose difference is the frustum.

Then
$$V = 2\pi a h^2 - 2\pi a h_1^2$$
$$= 2\pi a (h^2 - h_1^2)$$
$$= 2\pi a (h + h_1)(h - h_1).$$

But $R^2 = 4ah$, $r^2 = 4ah_1$,

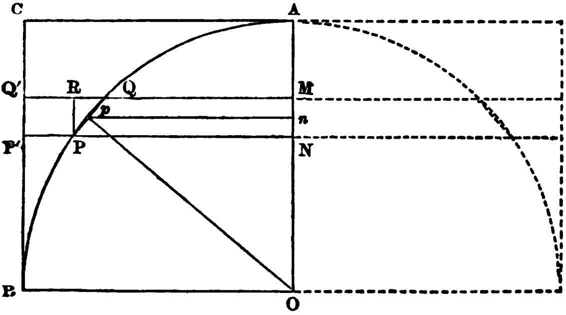
$$\therefore\ h + h_1 = \frac{1}{4a}(R^2 + r^2);$$

also $h - h_1 = k$.

Substituting these values of $(h + h_1)$ and $(h - h_1)$ in the expression for V, we obtain

$$V = 2\pi a \frac{1}{4a} k (R^2 + r^2) = \frac{\pi}{2} k (R^2 + r^2).$$

92. *To find the area of the surface of a sphere, and of any segment or zone of a sphere.*



Let PQ be one of the sides of a regular polygon inscribed in a circle of which O is the centre, and AB a quadrant.

Construct the square OC, touching the circle in the points B and A.

Through P, Q draw NPP', MQQ' parallel to OB, and PR parallel to OA; draw Op perpendicular to PQ, (and therefore bisecting it in p), and pn parallel to OB.

Now, if the whole figure revolve round OA fixed, the quadrant will generate a hemisphere, and the square will generate its circumscribing cylinder; also, the trapezoid MP will generate a frustum of a cone, and the parallelogram MP' a cylinder.

Now the surface generated by the chord PQ

$$=\pi . PQ\,(PN+QM)=2\pi . PQ . pn;$$

and the surface generated by $P'Q'=2\pi . NP' . P'Q'$

$$=2\pi . NP' . PR;$$

$$\therefore \frac{\text{surface generated by } PQ}{\text{surface generated by } P'Q'}=\frac{PQ . pn}{PR . NP'};$$

but the triangles QPR, Opn are similar,

$$\therefore \frac{PQ}{PR}=\frac{Op}{pn};$$

$$\therefore \frac{\text{surface generated by } PQ}{\text{surface generated by } P'Q'}=\frac{Op}{NP'}.$$

Now, in the limit, when the number of sides of the polygon is indefinitely increased, Op becomes the radius of the circle, and equal to NP',

$$\therefore\text{, ultimately, } \frac{\text{surface generated by } PQ}{\text{surface generated by } P'Q'}=\text{unity}.$$

Hence, in the sphere and in the cylinder, there can be inscribed the same number of figures, the surfaces of which are ultimately to each other in a ratio of equality;

∴ surface of sphere = surface of circumscribing cylinder.

Cor. 1. It will now be manifest that the surface of any segment or zone of a sphere is equal to the surface of the corresponding segment or zone of the circumscribing cylinder; for, in any two corresponding segments or zones, there can be inscribed the same number of figures, the surfaces of which are ultimately to each other in a ratio of equality.

COR. 2. If a be the radius of sphere, $2\pi a\,2a$ will be the surface of the circumscribing cylinder,

$$\therefore \text{ surface of sphere} = 4\pi a^2$$

$$= 4 \text{ area of generating circle.}$$

COR. 3. If h be the height of the segment generated by AQ,

$$h = AM = CQ',$$

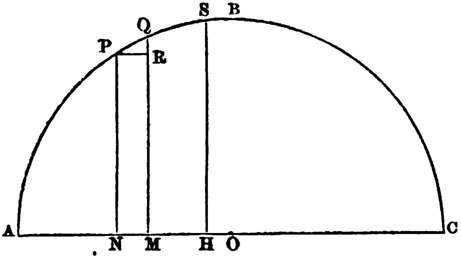
$$\therefore \text{ surface of segment} = 2\pi ah.$$

COR. 4. If k be the thickness of the zone generated by the arc PQ,

$$k = MN = P'Q',$$

$$\therefore \text{ surface of zone} = 2\pi ak.$$

93. *To find the volume of any segment of a sphere, and of a whole sphere.*



Let ABC be a semicircle, O its centre, and a its radius.

In AC take any point H, and draw HS perpendicular to AC.

Let AH be divided into n equal parts, of which NM is one, and AN the first r of these parts measured from A.

Through N, M draw NP, MQ parallel to HS, and through P draw PR parallel to AC, and join OP.

Let $\quad AH=h, \therefore NM=\frac{h}{n}$, and $AN=r\,\frac{h}{n}$.

Now, if the whole figure revolve round AC fixed, the semicircle will generate a sphere; and the rectangle MP will generate a cylinder, the volume of which will be

$$\pi \,.\, NP^2 \,.\, NM.$$

$$\begin{aligned} \text{But } NP^2 &= OP^2 - ON^2 = AO^2 - (AO - AN)^2 \\ &= AO^2 - AO^2 + 2AO\,.\,AN - AN^2 \\ &= 2AO\,.\,AN - AN^2; \end{aligned}$$

$$\therefore NP^2 = 2ar\,\frac{h}{n} - r^2\,\frac{h^2}{n^2}.$$

$$\begin{aligned} \therefore \text{ volume of cylinder} &= \pi\,\frac{h}{n}\left(2ar\,\frac{h}{n} - r^2\,\frac{h^2}{n^2}\right) \\ &= 2\pi a h^2\,\frac{r}{n^2} - \pi h^3\,\frac{r^2}{n^3}. \end{aligned}$$

Hence the volume of the segment of the sphere generated by the revolution of AHS round AH will be the limit of the expression

$$2\pi a\,\frac{h^2}{n^2}\,\{1+2+3+\ldots+(n-1)\}$$

$$-\pi\,\frac{h^3}{n^3}\,\{1^2+2^2+3^2+\ldots+(n-1)^2\},$$

when n is indefinitely increased.

$\therefore$ volume of segment $=\dfrac{2\pi a h^2}{2} - \dfrac{\pi h^3}{3}$, Art. 67, Cor.;

$$\therefore \text{ volume of segment} = \pi h^2\left(a - \frac{h}{3}\right).$$

Cor. 1. If $h=a$, the segment becomes a hemisphere;

$$\therefore \text{ volume of hemisphere} = \pi a^2\left(a-\frac{a}{3}\right)=\frac{2}{3}\pi a^3,$$

$$\therefore \text{ volume of sphere} = \frac{4}{3}\pi a^3.$$

This expression shows that the volume of a sphere is *two-thirds* of the volume of its circumscribing cylinder.

Cor. 2. To find the volume of any zone of a sphere it will be simpler to take the difference of two segments than to use a formula, except in the case of the *middle* zone of the sphere, contained between two parallel planes equidistant from the centre. In this case, if $2k$ be the thickness of the zone, the volume will be

$$2\left\{\frac{2}{3}\pi a^3-\pi(a-k)^2\left(a-\frac{a-k}{3}\right)\right\};$$

$$\therefore \text{ volume of middle zone} = 2\pi k\left(a^2-\frac{k^2}{3}\right).$$

94. *We may now find the volume of a spherical shell,—or the space included between two concentric spheres of unequal radii.*

Let R, r be the radii of the two spheres, and V the volume of the shell;

$$\begin{aligned}\therefore\ V&=\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3\\&=\frac{4}{3}\pi(R^3-r^3)\\&=\frac{4}{3}\pi(R-r)(R^2+Rr+r^2).\end{aligned}$$

If h be the thickness of the shell, $h=R-r$, and the expression may be written

$$V=\frac{4}{3}\pi R^2 h\left(1+\frac{r}{R}+\frac{r^2}{R^2}\right).$$

(1). Suppose the thickness very small compared with the radius R; then $\frac{r}{R}$ is nearly equal to unity, and we have, approximately,

$$V=4\pi R^2 h.$$

(2). Suppose the thickness considerable, and nearly equal to R; then $\frac{r}{R}$ is very small, and we have, approximately,

$$V=\frac{4}{3}\pi R^2 h.$$

These approximations are often very useful.

We may also find the volume of a *sector* of a spherical shell, bounded by any conical surface having its vertex at the common centre of the two spheres.

The student will not find it difficult to prove that the formula of Art. 41 will determine the volume of this solid.

95. *The volume of a sphere may easily be deduced from that of a cone, as follows:—*

If a cone and a hemisphere, of equal bases and altitudes, be placed with their axes parallel, and the vertex of the cone in the plane of the base of the hemisphere; and if the two solids, in this position, be cut by a plane perpendicular to their axes, the sum of the areas of the sections will be a *constant* area.

For, if x be the distance of the cutting plane from the vertex of the cone, the area of the section of the cone will be πx^2, and that of the hemisphere $\pi(a^2-x^2)$, (a being the radius), and the sum of these is πa^2, a constant area.

Hence the cone and hemisphere together are equal in volume to a *cylinder*, of which πa^2 is the area of the base, and a the altitude, that is, to the cylinder which circumscribes the hemisphere. Now the cone is *one-third* of this cylinder, therefore the hemisphere is *two-thirds* of the same.

Hence a sphere is $\frac{2}{3}$ of its circumscribing cylinder.

A sphere may also be considered the limit of an inscribed polyhedron, when the number of sides is increased indefinitely; from which it will be easy to show that the volume of a sphere $= \frac{1}{3}$ (area of surface) × radius.

Hence a sphere is equal in volume to a cone, the base of which is a *great circle* of the sphere, and the altitude twice the sphere's diameter.

96. *To find the volumes of the prolate and oblate spheroids, their segments and zones.*

If an ellipse revolve round its axis-major it will generate a solid of revolution called a *prolate spheroid*, and if it revolve round its axis-minor it will generate a solid called an *oblate spheroid.*

By a method analogous to that of Art. 75, the student will find no difficulty in proving that

$$\frac{\text{volume of prolate spheroid}}{\text{volume of sphere on axis-major}} = \frac{b^2}{a^2},$$

where a and b are the semi-axes major and minor of the ellipse; and also that this ratio is true of any two corresponding segments, or zones, of the spheroid and sphere.

Now volume of sphere on axis-major $= \frac{4}{3}\pi a^3$,

$$\therefore \text{ volume of prolate spheroid} = \frac{4}{3}\pi a b^2.$$

Also, volume of segment of spheroid

$$= \pi h^2 \frac{b^2}{a^2}\left(a - \frac{h}{3}\right),$$

h being the height of the segment.

And volume of middle zone of spheroid

$$= 2\pi \frac{b^2}{a^2} k \left(a^2 - \frac{k^2}{3}\right),$$

$2k$ being the thickness of the zone.

The corresponding expressions for the oblate spheroid will be—

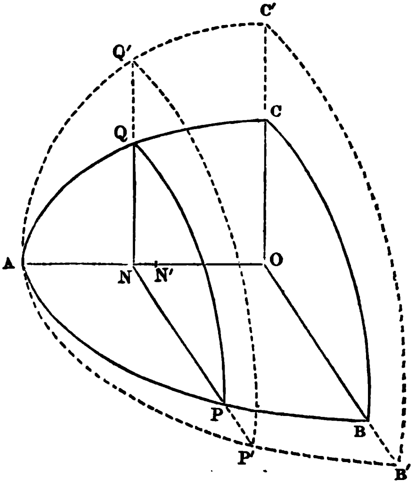
$$\text{volume of oblate spheroid} = \frac{4}{3}\pi a^2 b;$$

$$\text{volume of segment of spheroid} = \pi h^2 \frac{a^2}{b^2}\left(b - \frac{h}{3}\right);$$

$$\text{volume of middle zone of spheroid} = 2\pi \frac{a^2}{b^2} k \left(b^2 - \frac{k^2}{3}\right).$$

97. *To find the volume of an ellipsoid.*

An ellipsoid is generated in the following manner:—



Let *APB*, *AQC* be two elliptic quadrants having a common semi-axis-major *AO*, and their planes perpendicular to each other. Suppose the centre of a variable ellipse to move along *AO* with its plane always perpendicular to *AO*, and its vertices always in the curves *APB*,

AQC. The surface generated by this variable ellipse is called an ellipsoid, of which $ABC.O$ represents an octant.

Let $AB'C'.O$ be an octant of the sphere, on the major axis of the ellipsoid as diameter, contained between the same three planes AOB, BOC, COA.

Let $OA=a$, $OB=b$, $OC=c$; a being the greatest, and c the least of these.

We shall prove that the volume of the ellipsoid is equal to

$$\frac{4}{3}\pi abc.$$

Through any point N, in AO, draw a plane perpendicular to AO intersecting the ellipsoid in the ellipse of which PQ is a quadrant, and the sphere in the circle of which $P'Q'$ is a quadrant: and suppose a parallel plane to pass through a point N', near to M, and the two *inscribed* cylinders on these elliptic and circular bases to be completed;

$$\therefore \frac{\text{volume of elliptic cylinder}}{\text{volume of circular cylinder}} = \frac{\pi . NP . NQ . NN'}{\pi . NP'^2 . NN'}$$

$$= \frac{NP . NQ}{NP'^2};$$

but $$\frac{NP}{NP'} = \frac{OB}{OA}; \quad \frac{NQ}{NQ'} = \frac{OC}{OA}; \quad NP' = NQ';$$

$$\therefore \frac{NP . NQ}{NP'^2} = \frac{OB . OC}{OA^2};$$

$$\therefore \frac{\text{volume of elliptic cylinder}}{\text{volume of circular cylinder}} = \frac{OB . OC}{OA^2} = \frac{bc}{a^2}.$$

Hence, in the ellipsoid and in the sphere, there can be inscribed the same number of cylinders, the volumes of which are to each other always, and therefore ultimately, in the ratio $\frac{bc}{a^2}$; and their sums are ultimately the volumes of the ellipsoid and sphere.

Therefore, by Lemma IV.,

$$\frac{\text{volume of ellipsoid}}{\text{volume of sphere}}=\frac{bc}{a^2}.$$

Now volume of sphere $=\frac{4}{3}\pi a^3$;

$$\therefore \text{ volume of ellipsoid}=\frac{4}{3}\pi a^3\,\frac{bc}{a^2}=\frac{4}{3}\pi abc.$$

COR. 1. It will be manifest that any segment, or zone, of the ellipsoid, made by a plane or planes perpendicular to AO, will have to the corresponding segment, or zone, of the sphere the ratio $bc : a^2$.

COR. 2. The ellipsoid includes the two spheroids and the sphere as particular cases. Thus, if $b=c$, the ellipsoid becomes a prolate spheroid, (all sections perpendicular to OA being circles), and its volume $=\frac{4}{3}\pi ab^2$. If $b=a$, the solid becomes an oblate spheroid, (all sections perpendicular to OC being circles), and its volume $=\frac{4}{3}\pi a^2c$. If $b=c=a$, the ellipsoid becomes a sphere, and its volume $=\frac{4}{3}\pi a^3$.

These results agree with those already obtained for the spheroids and sphere.

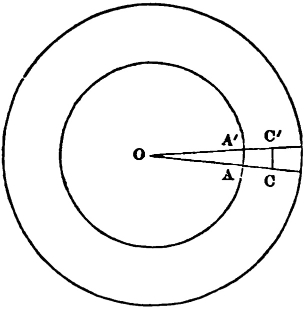
In the same manner we may prove that the ellipsoid is to the sphere of which OC is the radius, as $ab : c^2$; and to the sphere of which OB is radius, as $ac : b^2$.

98. *To find the volume of a circular ring, and the area of its surface.*

This solid is generated by the revolution of a circle about a straight line in its plane, but which does not meet it. This line is called the axis of the ring.

Let AB, $A'B'$ represent the edge of the generating circle in two positions very near to each other; and O the intersection of the axis with the plane of the paper. Bisect AB and $A'B'$ in C and C' respectively, and join CC'.

Then, ultimately, when the angle COC' is indefinitely diminished, CC' will become perpendicular to AB, and



the portion of the ring between AB and $A'B'$ will become a cylinder of which the generating circle will be the base, and CC' the altitude.

Let R be the mean radius of the ring $= OC$,

r the radius of the transverse section $= AC$,

and $\frac{2\pi}{n}$ the circular measure of the angle COC', so that the solid between AB and $A'B'$ will be the n^{th} part of the ring.

Then, ultimately,

$$\text{volume between } AB \text{ and } A'B' = \pi r^2\, 2R \sin\frac{\pi}{n},$$

$$\text{since } CC' = 2OC' \sin\frac{COC'}{2} = 2R\sin\frac{\pi}{n};$$

$$\therefore \text{ volume of ring} = n\, 2\pi r^2\, R \sin\frac{\pi}{n},$$

when n is indefinitely increased,

$$\text{i.e. volume of ring} = \text{limit of } 2\pi^2 r^2 R \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}};$$

$$\text{and since the limit of } \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 1,$$

$$\therefore \text{ volume of ring} = 2\pi^2 r^2 R.$$

We may write this formula thus: $V = \pi r^2 \times 2\pi R$, which shows that *the volume of the ring is equal to the area of the generating circle multiplied by the length of the path described by its centre;* and it will be manifest that this result is true for any segment of the ring, the bounding planes of which pass through the axis. Thus, if θ be the circular measure of the angle subtended at the centre of the ring by any segment of it, the volume of this segment will be

$$\pi r^2 \times R\theta.$$

The student will now easily prove that the volume of the solid generated by the motion of a circle, the centre of which moves along any *continuous* curve, with its plane *always perpendicular to the tangent*, will be found by multiplying the area of the generating circle by the length of the path described by its centre.

Cor. The same formula which we have proved for the volume of a circular ring when the section is a circle, will also determine the volume of a circular ring, the transverse section of which is an ellipse, supposing one of the axes of the ellipse to be *perpendicular to the plane of its motion*. The formula will also determine the volume of a ring generated by the revolution of any closed curve, provided the curve be symmetrical about a line through C, perpendicular to the plane of motion.

The area of the surface of the ring may also be found in a similar manner.

For the area of the surface of the ultimate cylinder $ABB'A'$

$$= 2\pi r \times CC' = 2\pi r \; 2R \sin\frac{\pi}{n};$$

$\therefore$ area of surface of ring $=4n\pi Rr\sin\frac{\pi}{n}$, ultimately,

$$=4\pi^2 Rr\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\text{, ultimately;}$$

$\therefore$ area of surface of ring $=4\pi^2 Rr=2\pi r\times 2\pi R$.

Hence the area of the surface of the ring is equal to the circumference of the generating circle, multiplied by the length of the path described by its centre.

This result will also be true for the surface of any solid generated as before explained.

99. *To find the volume of a spiral spring, the transverse section of which is a circle.*

This solid may be generated by the motion of a circle, the centre of which moves along the curve called the Helix, with its plane always perpendicular to the tangent.

Hence, by the last Art., the volume will be found by multiplying the area of the generating circle by the length of the helix.

Now the helix is the curve traced upon the surface of a circular cylinder by a point, the direction of whose motion makes a *constant* angle with the generating line of the cylinder.

Let a be this constant angle, r the radius of the cylinder, and θ the *horizontal* angle described by the generating point in rising *vertically* through the distance h it will easily be seen that

$$\frac{h}{r\theta}=\cot a,$$

and $\therefore h=r\theta\cot a$.

Now the length of the helix described will manifestly be

$$\sqrt{(h^2 + r^2\theta^2)},$$

h and $r\theta$ being the two sides of a right-angled triangle, the hypothenuse of which is the length of the curve.

$$\begin{aligned} \therefore \text{ length of helix} &= \sqrt{(r^2\theta^2 \cot^2 \alpha + r^2\theta^2)} \\ &= r\theta \sqrt{(1 + \cot^2 \alpha)} \\ &= r\theta \operatorname{cosec} \alpha. \end{aligned}$$

Hence the volume of the corresponding portion of the spiral spring will be

$$\pi a^2 r\theta \operatorname{cosec} \alpha,$$

a being the radius of the generating circle.

Again, since $\qquad r\theta = h \tan \alpha,$

this expression becomes

$$\pi a^2 h \operatorname{cosec} \alpha \tan \alpha = \frac{\pi a^2 h}{\cos \alpha}.$$

Hence the volume will be known when we know a and α, and the vertical height h, of the spring. This quantity h is easily measured.

We may also obtain an easy formula for the volume of the spring, when we know the area of its section, made by a plane perpendicular to the axis of the cylinder.

Let A be this area; then, by the relation proved in Art. 38, Cor. 2, extended to curvilinear areas,

$$\pi a^2 = A \cos \alpha; \quad \therefore A = \frac{\pi a^2}{\cos \alpha}.$$

Hence, by substitution, we have

$$\text{volume of spring} = Ah.$$

The section, of which A is the area, will be an ellipse, and $2a$ will be its minor axis: when the major axis can be *measured*, the area can at once be found.

Cor. The volume of one complete coil of the spring will be found by making $\theta=2\pi$, and $\therefore h=2\pi r\cot a$;

$$\therefore \text{ volume of complete coil } =2\pi^2a^2r\operatorname{cosec} a,$$

in terms of r.

Also, if h' be the vertical distance between two successive coils, or their distance parallel to the axis of the cylinder;

$$\therefore \text{ volume of complete coil}=\frac{\pi a^2h'}{\cos a}.$$

The spiral spring is a solid of great importance in its practical applications.

Ex. A spiral spring consists of 9 complete coils; its vertical height is 17 inches, the diameter of its transverse section 1 inch, and the mean radius of the spiral 8 inches; and the solid is bounded by two horizontal planes: find its volume.

It will be obvious that the volume will be the same as if the solid were bounded by two parallel planes perpendicular to the tangents at the extremities of the helix described by the centre of the generating circle.

Therefore the formula

$$V=\frac{\pi a^2h}{\cos a}$$

will determine the volume.

Here $h=17$, $a=\frac{1}{2}$, $\therefore \pi a^2=\frac{\pi}{4}=0.7854$;

$$\cot a=\frac{h}{9\times 2\pi 4}=\frac{17}{72\pi},$$

$$\therefore \frac{1}{\cos a}=\sqrt{1+\tan^2a}=\sqrt{1+\left(\frac{72\pi}{17}\right)^2}=13.34.$$

Hence $V=17\times 0.7854\times 13.34=174.059$ cubic inches.

CHAPTER IX.

THE "PRISMOIDAL FORMULA," AND ITS APPLICATIONS.

100. In Chapter V., Art. 46—49, we investigated the "Prismoidal Formula" for determining the volumes of certain solids, bounded by plane rectilinear figures. The formula, it will be remembered, is as follows:—

$$6V = l(A_1 + A_2 + 4A),$$

where V is the volume of the solid, l its length, A_1, A_2 the areas of the two parallel ends, and A the area of the mean section,—or section midway between the ends and parallel to them.

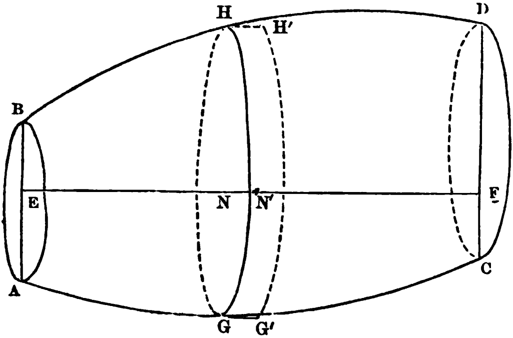
In the present Article, we propose to show that this formula is sufficient to determine the volumes of all solids which are contained between two parallel planes, and which have this property:—that the area of any section, parallel to these planes, can be expressed by a rational integral algebraical function, of a degree not higher than the third, of its distance from either of these bounding planes; or if A_x be the area of a section at a distance x from one end of the solid, then

$$A_x = A' + Bx + Cx^2 + Dx^3,$$

where A', B, C, D are constant for the same solid.

These coefficients may be either positive or negative, and any one, two, or three of them may be zero.

Let $ACDB$ be a solid contained between two parallel planes AB and CD; and let EF be any straight line perpendicular to these planes.



Take any point N in EF, and through it draw a plane GH perpendicular to EF. Let $EN=x$, and let A_x be the area of the section GH. Then, if the solid have the property which we have stated,

$$A_x = A' + Bx + Cx^2 + Dx^3.$$

Divide EF into n equal parts, and let EN be the first r of these parts, and NN' one of them.

Then, if $EF=h$, $NN'=\frac{h}{n}$, and $EN=r\frac{h}{n}$.

Through N' draw a plane parallel to GH, and complete the cylindrical solid $GH\,H'G'$, as shown in the figure.

The volume of this solid is

$$A_x \times NN' = (A' + Bx + Cx^2 + Dx^3)\frac{h}{n},$$

or $$\text{Volume} = \left(A' + Br\frac{h}{n} + Cr^2\frac{h^2}{n^2} + Dr^3\frac{h^3}{n^3}\right)\frac{h}{n},$$

since $x = r\frac{h}{n}$;

$$\therefore \text{Volume} = A'\frac{h}{n} + Br\frac{h^2}{n^2} + Cr^2\frac{h^3}{n^3} + Dr^3\frac{h^4}{n^4}.$$

Hence the sum of the volumes of all the inscribed cylindrical solids will be expressed by

$$A'\frac{h}{n}(1+1+1+\ldots\text{to } n \text{ terms}) + B\frac{h^2}{n^2}(1+2+3+\ldots+n)$$

$$+C\frac{h^3}{n^3}(1^2+2^2+3^2+\ldots+n^2) + D\frac{h^4}{n^4}(1^3+2^3+3^3+\ldots+n^3);$$

and the volume of the proposed solid will be the limit of this expression when n is indefinitely increased.

Now the limit of $\frac{1}{n}(1+1+1+\ldots\text{to } n \text{ terms}) = 1$;

the limit of $\frac{1}{n^2}(1+2+3+\ldots\ldots\ldots+n) = \frac{1}{2}$,

the limit of $\frac{1}{n^3}(1^2+2^2+3^2+\ldots\ldots+n^2) = \frac{1}{3}$,

and the limit of $\frac{1}{n^4}(1^3+2^3+3^3+\ldots\ldots+n^3) = \frac{1}{4}$,

as proved in Art. 67.

Hence, by substituting these limiting values, we obtain

$$\text{Volume of solid} = A'h + B\frac{h^2}{2} + C\frac{h^3}{3} + D\frac{h^4}{4};$$

and therefore, if V be the volume,

$$6V = 6A'h + 3Bh^2 + 2Ch^3 + \frac{3}{2}Dh^4.$$

Now A_1 is the value of $A' + Bx + Cx^2 + Dx^3$ when $x = 0$, A_2 its value when $x = h$, and A its value when $x = \frac{h}{2}$;

$$\therefore \quad A_1 = A';$$

$$A_2 = A' + Bh + Ch^2 + Dh^3;$$

$$A = A' + B\frac{h}{2} + C\frac{h^2}{4} + D\frac{h^3}{8},$$

or
$$4A = 4A' + 2Bh + Ch^2 + \frac{1}{2}Dh^3.$$

Adding together these values of A_1, A_2, and $4A$, we have

$$A_1 + A_2 + 4A = 6A' + 3Bh + 2Ch^2 + \frac{3}{2}Dh^3;$$

$$\therefore h(A_1 + A_2 + 4A) = 6A'h + 3Bh^2 + 2Ch^3 + \frac{3}{2}Dh^4.$$

But the right-hand member is identical with the expression already found for $6V$;

$$\therefore \ 6V = h(A_1 + A_2 + 4A).$$

Remark. The student should observe that since
$$A' + Bx + Cx^2 + Dx^3$$
represents an *area*, each of its terms must represent an *area*, and must therefore be a quantity of *two dimensions*.

Now x represents a line, and is therefore of *one* dimension. Hence the dimensions of A' are *two*, of B *one*, of C *zero*, and of D *minus one;* i.e. A' represents an *area*, B a *line*, C a *number*, and D a quantity of the form $\frac{1}{a}$ where a represents a *line.* Attention to these considerations will frequently detect errors, and sometimes prevent them.

101. In this and the following Article we shall examine some of the more important solids to which the Prismoidal Formula is applicable for the determination of their volumes.

The general expression for A_x, the area of a transverse section at a distance x from one end of the solid, being
$$A' + Bx + Cx^2 + Dx^3,$$

there will be as many possible varieties as there are combinations of four things taken *one*, *two*, *three*, and *four* together; i.e. there will be $2^4 - 1$, or 15 varieties.

Corresponding to each of these there will be *at least one solid of revolution*, generated by the curve whose equation is, in the general case,

$$\pi y^2 = A' + Bx + Cx^2 + Dx^3;$$

for, if x, y be the coordinates of any point in the curve, y being the revolving ordinate of any point, πy^2 will be the area of the section at distance x from one end of the solid,—supposing x to be measured from that end.

We shall examine, first, the simpler cases of the solids of revolution, and afterwards, those solids which require a different mode of generation.

(1). Let $\pi y^2 = A'$; $\therefore y$ is constant, and the solid is a circular cylinder.

(2). Let $\pi y^2 = Bx$; $\therefore y^2 \propto x$, and the solid is a paraboloid of revolution; for, in a parabola, the square of the ordinate varies as the abscissa.

(3). Let $\pi y^2 = Cx^2$; $\therefore y \propto x$, and the solid is a right circular cone; for, in a straight line, which, by its revolution round another straight line, generates a cone, the ordinate varies as the abscissa.

(4). Let $\pi y^2 = Dx^3$; $\therefore y^2 \propto x^3$, and the solid is a semi-cubical paraboloid of revolution, or the solid generated by the revolution round the axis of x of the curve called the semi-cubical parabola; for, in this curve, the square of the ordinate varies as the cube of the abscissa.

The student will find no difficulty in tracing the form of this curve, which consists of two branches symmetrical about the axis of x, and *convex* to it, instead of being *concave* to that axis as in the common parabola. The two branches terminate in a *cusp* at the origin.

(5). Let $\pi y^2 = A' + Bx$; $\therefore y^2 \propto (h + x)$, where h is constant, and the solid will be a *frustum* of a paraboloid of revolution, h being the length of the segment cut off.

(6). Let $\pi y^2 = A' + Cx^2$; supposing A' and C positive, this will be the equation to a hyperbola, the minor axis being the axis of x, and the centre the origin; or, to its conjugate hyperbola, when the major axis is the axis of x.

For, in the first case, if x, y be the coordinates of any point in the hyperbola, and a, b its semi-axes,

$$x^2 = \frac{b^2}{a^2}(y^2 - a^2),$$

$$\therefore\ y^2 = a^2 + \frac{a^2}{b^2}x^2.$$

Comparing this with the given equation

$$y^2 = \frac{A'}{\pi} + \frac{C}{\pi}x^2,$$

we see that A' and C are positive, and $\frac{C}{\pi} > 1$.

For the conjugate hyperbola, the equation will be

$$y^2 = b^2 + \frac{b^2}{a^2}x^2,$$

and therefore, in this case, A' and C are positive, $\frac{C}{\pi} < 1$.

Hence the solid will be that generated by the revolution of a hyperbola round its minor axis when C is $> \pi$, or, by the revolution of its conjugate round its major axis when $C < \pi$. These solids may be called oblate hyperboloids.

(7). Let $\pi y^2 = A' + Dx^3$. In this case the solid is generated by the revolution of a curve, somewhat similar in form to the semi-cubical parabola, round a line parallel to the axis of x, and at a constant distance from it.

(8). Let $\pi y^2 = Bx + Cx^2$. In this case the solid may be a sphere, a prolate spheroid, an oblate spheroid, a hyperboloid of revolution, or its conjugate hyperboloid.

For the sphere and spheroids B will be positive and C negative; and for the hyperboloid B and C will both be positive. Also, for the sphere $\frac{C}{\pi}=1$, for the prolate spheroid and the hyperboloid $\frac{C}{\pi}<1$, and for the oblate spheroid and the conjugate hyperboloid $\frac{C}{\pi}>1$.

Thus, in the figure of Art. 93, we have, for the sphere,

$$NP^2=2AO\,.\,AN-AN^2;$$

and putting x, y for AN, NP, and a for the radius of the circle, this becomes

$$y^2=2ax-x^2.$$

Comparing this equation with the given equation

$$y^2=\frac{B}{\pi}x+\frac{C}{\pi}x^2,$$

we see that B is positive and C negative, and $\frac{C}{\pi}=1$.

Again, in the figure of Art. 75, we have, for the prolate spheroid,

$$NP=\frac{BC}{AC}NP';$$

or, putting x, y for $A'N$, NP, y' for NP', and a, b for the semi-axes of the ellipse,

$$y^2=\frac{b^2}{a^2}y'^2=\frac{b^2}{a^2}(2ax-x^2).$$

And, comparing this equation with the given equation, we see that B is positive and C negative, and $\frac{C}{\pi}<1$.

For the oblate spheroid, we shall have

$$y^2=\frac{a^2}{b^2}(2bx-x^2),$$

and therefore B will be positive and C negative, in this case, and $\frac{C}{\pi} > 1$.

For the hyperboloids the only difference will be that the coefficient of x^2 in the last two equations will be positive instead of negative. This will make C positive as well as B; and the solid will be a hyperboloid of revolution or its conjugate hyperboloid, according as $\frac{C}{\pi}$ is $<$ or > 1; and a rectangular hyperboloid when $\frac{C}{\pi} = 1$.

(9). Let $\pi y^2 = A' + Bx + Cx^2$. In this case the solid will be a *frustum* of a circular cone, or of the sphere, spheroids, or hyperboloids of revolution, made by planes perpendicular to the axis.

In the frustum of the cone A', B, C are all positive. The other solids will be distinguished, as in the last case, by the values and signs of the constants B and C.

(10). Let $\pi y^2 = A' + Bx + Cx^2 + Dx^3$. In this case the solid is a *frustum* of a semi-cubical paraboloid of revolution. For, if x be the distance of the section from the smaller end of the frustum, and h the length of the segment cut off,

$$\therefore\ y^2 \propto (h+x)^3,$$

$$\therefore\ \pi y^2 \textit{ is of the form } A' + Bx + Cx^2 + Dx^3.$$

We have now examined ten of the fifteen possible varieties of the general expression for the area of the section when the solid is one of revolution; the other five belong to solids generated by curves too complex to be discussed with advantage in this Treatise.

102. We shall now examine some of the solids to which the Prismoidal Formula is applicable, but which are not solids of revolution.

(1). Let $A_x = A'$. In this case all the transverse sections are constant. This is the property of all prisms, and cylinders; also, of all solids formed by the *twisting* of prisms and cylinders, supposing the twisting to be uniform throughout the length of the solid. For example, the square-threaded screw, the rifled barrel of a gun, and the twisted columns sometimes seen in Gothic Architecture.

(2). Let $A_x = Bx$. This is a property of the elliptic paraboloid, or the solid generated by the motion of a variable ellipse, the axes of which are the double ordinates of two parabolas which have a common axis and a common vertex, the plane of the ellipse being always perpendicular to this axis; for, in this solid, the area of the section at distance x from the vertex will be $\pi yy'$, where y and y' are the ordinates of the two parabolas, and since both y^2 and y'^2 vary as x, $\therefore$ $\pi yy'$ varies as x.

Thus, in the figure of Art. 97, if we suppose AB and AC to be *parabolas* which have a common vertex A, and a common axis AO, and PNQ to be a quadrant of the generating ellipse, NP^2 and NQ^2 will both vary as AN, and therefore $\pi . NP . NQ$ will vary as AN.

(3). Let $A_x = Cx^2$. This is a property of all pyramids and cones, whatever may be their bases. In the former, all parallel transverse sections are similar rectilinear figures, and in the latter, similar curvilinear figures; and, in each case, the area varies as the square of its distance from the vertex.

(4). Let $A_x = Dx^3$. The solid, in this case, will be an elliptic semi-cubical paraboloid. This may easily be seen by substituting semi-cubical parabolas for the common parabolas in (2).

(5). Let $A_x = A' + Bx$. This is manifestly a property of a *frustum* of an elliptic paraboloid, as may be seen by comparing it with (2).

(6). Let $A_x = A' + Cx^2$. This is a property of a *groin*, a simple case of which is the square groin seen in the vaults of large buildings.

This solid may be generated by a variable square, which moves, parallel to itself, with the middle points of two opposite sides always in a semicircle, the plane of which is perpendicular to that of the square.

If a be the radius of the semicircle, y the length of a side of the square when at a distance x from the centre of the semicircle;

$$\therefore\ y^2 = 4(a^2 - x^2).$$

Hence, in this solid, C will be negative; and A' will represent the area of the base.

(7). Let $A_x = Bx + Cx^2$.

This is a property of the ellipsoid, and of the elliptic hyperboloid.

If we refer to the figure of Art. 97, in which AB, AC are ellipses, and PNQ a quadrant of the variable ellipse which generates the ellipsoid, we have, for the area of this ellipse, $\pi . NP . NQ$.

Let $OA = a$, $OB = b$, $OC = c$, $AN = x$, $NP = y$, $NQ = y'$;

$$\therefore\ y^2 = \frac{b^2}{a^2}(2ax - x^2), \quad y'^2 = \frac{c^2}{a^2}(2ax - x^2);$$

$$\therefore\ \text{area of section} = \pi y y' = \pi \frac{bc}{a^2}(2ax - x^2).$$

Hence, for the ellipsoid, B is positive and C negative.

For the elliptic hyperboloid, the equations will be

$$y^2 = \frac{b^2}{a^2}(2ax + x^2), \quad y'^2 = \frac{c^2}{a^2}(2ax + x^2);$$

$$\therefore\ \pi y y' = \pi \frac{bc}{a^2}(2ax + x^2).$$

Hence, for this solid, in which AB, AC are hyperbolas, B and C are both positive.

(8). Let $A_x = A' + Bx + Cx^2$. This is a property of a prismoid; and of any frustum of a pyramid or cone, whatever may be the base: also, of any frustum of an ellipsoid, or elliptic hyperboloid, made by planes perpendicular to the axis of the hyperboloid, or to any one of the three axes of the ellipsoid. For the prismoid this will appear from the fig. on page 78, if we suppose $KLMN$ to be a section at distance x (OE) from the smaller end; for then $OPNK$ is a constant area, the area of $OPVU$ varies as x, and those of UOL, VPM vary as x^2.

For the other solids, if x be the distance of the section from one end of the frustum, and h the length of the segment cut off, A_x will vary as $(h+x)^2$ in the pyramids and cones, and as

$$\{P(h+x)+Q(h+x)^2\}$$

in the ellipsoid, and elliptic hyperboloid, P and Q being constants;

$\therefore$ A_x may be expressed, in all these cases, by

$$A'+Bx+Cx^2.$$

For the prismoid, pyramids, and cones, A', B, C are all positive: for the ellipsoid B is positive and C negative; and for the elliptic hyperboloid, B and C are both positive.

103. There are also many other solids to which the Prismoidal Formula is applicable. In any case which may occur, it is only necessary to determine whether the parallel sections can be expressed by

$$A'+Bx+Cx^2+Dx^3,$$

n which x is the distance of the section from one end of the solid, and A', B, C, D constants which may be positive or negative, and any one, two, or three of them may be zero.

Taking the general case, if h be the length of the solid, we have

$$A_1=A',$$

$$A_2=A'+Bh+Ch^2+Dh^3,$$

$$4A=4A'+2Bh+Ch^2+\frac{1}{2}Dh^3;$$

and by substituting these values of A_1, A_2, and $4A$ in the formula

$$6V=h(A_1+A_2+4A),$$

we obtain the volume of the solid.

104. Examples to illustrate the Prismoidal Formula.

Ex. 1. To find the volume of a sphere.

Let a be the radius of the sphere;

$$\therefore\ h=2a,\quad A_1=0,\quad A_2=0,\quad 4A=4\pi a^2;$$

$$\therefore\ 6V=2a4\pi a^2=8\pi a^3;$$

$$\therefore\ V=\frac{4}{3}\pi a^3.$$

Ex. 2. To find the volume of an ellipsoid.

Let a, b, c be the three semi-axes, a the greatest;

$$\therefore\ h=2a,\quad A_1=0,\quad A_2=0,\quad 4A=4\pi bc;$$

$$\therefore\ 6V=2a4\pi bc=8\pi abc;$$

$$\therefore\ V=\frac{4}{3}\pi abc.$$

Ex. 3. A segment of a paraboloid being cut off by a plane perpendicular to the axis, it is required to bisect it by a parallel plane.

Let h be the height of the segment, k the radius of its base, and a the latus-rectum of the generating parabola.

Then in the formula

$$6V=h(A_1+A_2+4A),$$

$$A_1=0,\ A_2=\pi k^2=\pi ah,\ 4A=4\pi a\frac{h}{2};$$

$$\therefore\ 6V=\pi h(ah+2ah)=3\pi ah^2.$$

Similarly, if x be the height, and V' the volume, of the segment cut off by the bisecting plane,

$$6V'=3\pi ax^2;$$

$$\text{but } V=2V',\quad \therefore\ 3\pi ah^2=6\pi ax^2;$$

$$\therefore\ h^2=2x^2,\quad \text{and } \therefore\ x=\frac{h}{\sqrt{2}}.$$

Hence, if a square be constructed on h as diagonal, x will be the length of one side.

Ex. 4. A spherical zone is 4 feet thick, and the diameters of its opposite faces are 12 and 8 feet; find the volume.

Let y be the radius of the mean section, x its distance from the centre of the sphere from which the zone is cut, and r the radius of the sphere.

Then $A_1 = \pi 4^2$, $A_2 = \pi 6^2$, $4A = 4\pi y^2$, and $h = 4$;

$$\therefore\ 6V = 4\pi\,(4^2 + 6^2 + 4y^2);$$

and we have to find y^2.

Now, by the geometry of the sphere,

$$y^2 = r^2 - x^2;\quad 6^2 = r^2 - (x-2)^2;\quad 4^2 = r^2 - (x+2)^2;$$

$$\therefore\ y^2 - 6^2 = (x-2)^2 - x^2 = -2\,(2x-2) = -4x + 4;$$

and

$$y^2 - 4^2 = (x+2)^2 - x^2 = 2\,(2x+2) = 4x + 4;$$

$$\therefore\ 2y^2 - 36 - 16 = 8, \text{ or } y^2 = 30.$$

Hence

$$6V = 4\pi\,(16 + 36 + 120);$$

$$\therefore\ V = \frac{344}{3}\pi = \frac{344}{3} \times \frac{355}{113} = 360.23;$$

$\therefore$ volume of zone $= 360$ cubic feet 33 cubic inches.

Ex. 5. To divide a prolate spheroid into three equal parts by planes perpendicular to the axis.

It is obvious that these planes will be equidistant from the centre, one on either side of it. Let x be this distance, and let a, b be the semi-axes of the generating ellipse, and V the volume of the middle zone intercepted between the cutting planes.

Then the area of the section made by either plane will be

$$\pi\,\frac{b^2}{a^2}\,(a^2 - x^2),$$

$$\therefore\ A_1 = A_2 = \pi\,\frac{b^2}{a^2}\,(a^2 - x^2);\quad \text{and } 4A = 4\pi b^2;$$

Substituting these values of A_1, A_2, and $4A$ in the formula

$$6V = h\,(A_1 + A_2 + 4A);$$

we obtain

$$6V = 2x\left\{2\pi\frac{b^2}{a^2}(a^2 - x^2) + 4\pi b^2\right\}$$

$$\therefore\ 6V = 4\pi b^2\left(3x - \frac{x^3}{a^2}\right).$$

Again, if V' be the volume of the spheroid,

$$6V' = 8\pi b^2 a;$$

and $V' = 3V$, by the conditions of the problem;

$$\therefore\ 8\pi b^2 a = 12\pi b^2\left(3x - \frac{x^3}{2}\right),$$

$$\therefore\ 2a = 3\left(3x - \frac{x^3}{a^2}\right);$$

$$\therefore\ 3x^3 - 9a^2x + 2a^3 = 0,$$

$$\therefore\ \left(\frac{x}{a}\right)^3 - 3\left(\frac{x}{a}\right) + \frac{2}{3} = 0.$$

This is a cubic equation which has all its roots real;

for, when $\frac{x}{a} = 0$, the left-hand member is positive,

when $\frac{x}{a} = 1$, the left-hand member is negative,

and when $\frac{x}{a} = 2$, the left-hand member is positive again.

Hence the equation has a real positive root between 0 and 1, and another between 1 and 2; also it has one real negative root because its last term is positive.

It is manifest that the only root which is admissible in the problem is the positive root between 0 and 1.

This root may be found by approximation to be 0·22607, as far as 5 places of decimals;

$$\therefore\ \frac{x}{a} = \cdot 22607,\ \text{or}\ x = a \times \cdot 22607,\ = \frac{9}{40}a,\ \text{approximately.}$$

It should be observed that the equation is independent of the minor axis of the generating ellipse; hence all spheroids which have the same major axis are trisected by the same two planes.

Ex. 6. To show that the volume of the solid generated by the revolution, round the minor axis, of an arc of a hyperbola, cut off by a chord equal and parallel to the minor axis, is twice the volume of the oblate spheroid generated by the revolution of the ellipse which has the same axes.

Let $2a$, $2b$ be the axes of the hyperbola, $2b$ being the length of the minor axis; and let x, y be the coordinates of any point in the curve referred to the axes of the curve as coordinate axes, the minor axis being that of x.

Then we have

$$y^2 = \frac{a^2}{b^2}(b^2 + x^2),$$

and

$$6V = h(A_1 + A_2 + 4A).$$

Here $h = 2b$, the minor axis being the length of the solid;

A_1 is the value of πy^2 when $x = b$, $\therefore\ A_1 = 2\pi a^2$;

A_2 is the value of πy^2 when $x = -b$, $\therefore\ A_2 = 2\pi a^2$;

A is the value of πy^2 when $x = 0$, $\therefore\ 4A = 4\pi a^2$;

$$\therefore\ 6V = 2b\,(8\pi a^2) = 16\pi a^2 b.$$

Again, for the oblate spheroid,

$$A_1 = 0,\ A_2 = 0,\ 4A = 4\pi a^2, \text{ and } h = 2b.$$

Hence, if V' be its volume,

$$6V' = 2b\,(4\pi a^2) = 8\pi a^2 b.$$

Therefore

$$V = 2V';$$

or the oblate hyperboloid is double of the oblate spheroid.

CHAPTER X.

ON THE PARABOLIC SPINDLE; AND ITS APPLICATION TO THE GAUGING OF CASKS.

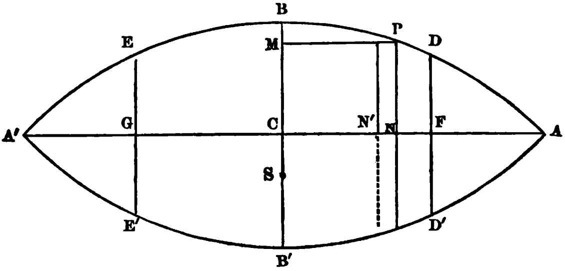
105. A SPINDLE is the name given to a solid generated by the revolution of an arc of a curve round its chord,—the curve being supposed symmetrical about a straight line perpendicular to the chord, through its middle point.

Thus we may have a circular spindle, generated by the revolution of a circular arc round its chord; an elliptic spindle, generated by the revolution of an elliptic arc round a chord parallel to either axis; or, a parabolic spindle, by the revolution of a parabolic arc round a chord perpendicular to the axis.

We cannot find the volumes of the circular or elliptic spindles without the aid of the Integral Calculus, and, indeed, the formulæ in these cases are too complicated to be of any value in practice; but the volume of the parabolic spindle may be found without difficulty, and this solid happens to be of more importance than the others in its practical applications.

106. *To find the volume of a parabolic spindle; and of any frustum of the spindle, made by planes perpendicular to the axis.*

Let ABA' be an arc of a parabola, B its vertex, S its focus, and AA' a double ordinate perpendicular to the axis; and suppose the figure to revolve round AA' fixed. The solid generated will be a parabolic spindle.



We shall first find an expression for the volume of the frustum $BB'D'D$ contained between the two planes BB' and DD', of which BB' passes through the vertex of the parabolic arc.

Let $CF=k$, $CA=h$, $CB=R$, $FD=r$;

and let a be the latus-rectum of the parabola $=4BS$.

Divide CF into n equal parts, and let CN be the first p of these measured from C, and NN' one of them;

$$\therefore NN'=\frac{k}{n}, \quad \text{and} \quad CN=p\frac{k}{n}.$$

Through N draw NP parallel to CB, and through P draw PM parallel to CA, and complete the rectangle PN' as in the figure.

When the figure revolves round AA', this rectangle will generate a cylinder, the volume of which will be

$$\pi . NP^2 \times NN'.$$

Let $BM=x$, $MP=y$; then, since the curve BPA is a parabola,

$$y^2=ax.$$

Also $$NP = CM = CB - BM = R - x;$$

$$\therefore\ NP = R - \frac{y^2}{a};$$

$$\therefore\ NP^2 = R^2 - 2R\frac{y^2}{a} + \frac{y^4}{a^2}$$

or $$NP^2 = R^2 - 2R\frac{p^2k^2}{an^2} + \frac{p^4k^4}{a^2n^4},$$

since $y = p\dfrac{k}{n}$;

∴ volume of cylinder generated by PN'

$$= \pi\left(R^2\frac{k}{n} - 2R\frac{p^2k^3}{an^3} + \frac{p^4k^5}{a^2n^5}\right).$$

Hence the volume of the frustum will be found by giving to p the values 1, 2, 3, …… n, and taking the limit of the sum of the resulting expressions;

$$\therefore\ \text{volume of frustum} = \pi k\left(R^2 - \frac{2}{3}R\frac{k^2}{a} + \frac{1}{5}\frac{k^4}{a^2}\right);$$

but, since BA is a parabola, $k^2 = a(R - r)$,

∴ volume of frustum

$$= \pi k\left\{R^2 - \frac{2}{3}R(R - r) + \frac{1}{5}(R - r)^2\right\}.$$

Hence we find, by reduction,

$$\text{Volume of frustum} = \pi k\left(\frac{8}{15}R^2 + \frac{4}{15}Rr + \frac{1}{5}r^2\right).$$

This gives the volume in terms of the length and the radii of the ends.

To find the volume of the spindle we have only to make $k = h$, then $r = 0$, and the solid will become half the spindle;

$$\therefore\ \text{Volume of spindle} = \frac{8}{15}2\pi R^2h.$$

But $2\pi R^2h$ is the volume of the circumscribing cylinder;

$\therefore$ volume of spindle $= \frac{8}{15}$ (circumscribing cylinder).

Again, if we make $CG = CF$, and through G draw a plane EE' perpendicular to the axis, (and therefore parallel to the plane DD'), the solid contained between these two planes is called a *middle frustum* of the spindle.

Its volume will be twice the volume of the frustum $BB'D'D$;

$\therefore$ volume of middle frustum of spindle

$$= 2\pi k\left(\frac{8}{15}R^2 + \frac{4}{15}Rr + \frac{1}{5}r^2\right).$$

This solid is of great importance in the gauging of casks; for the middle frustum of a parabolic spindle is a very close approximation to the general form of a cask.

107. *To gauge a cask; that is, to determine, approximately, the capacity of a cask from certain measurements of it.*

In the figure of the last Art. suppose $EE'D'D$ to represent a cask; EE', DD' being the *head* diameters, BB' the *bung* diameter, and FG the length.

These dimensions are the only quantities required to determine, approximately, the capacity of the cask.

There are four forms of solids usually employed in gauging casks, which are as follows:—

(1) The middle frustum of a prolate spheroid;

(2) The middle frustum of a parabolic spindle;

(3) The double frustum of two equal paraboloids;

(4) The double frustum of two equal cones:—

the last two varieties being formed by two equal symmetrical solids placed base to base.

It is probable that no cask has so great a capacity as the first of these forms, and that very few, if any, have so great a capacity as the second, or so small a capacity as the third; and certainly none are so small as the last variety.

We shall first obtain expressions for the volumes of each of the above solids, and afterwards determine the greatest error which can be made by assuming any one of them to be the *true* form of the cask.

(1). Suppose $EE'D'D$ to be the middle frustum of a prolate spheroid, V_1 its volume. The Prismoidal Formula is applicable to this solid, and from it we obtain

$$V_1 = \frac{2\pi k}{3}(2R^2 + r^2).$$

(2). Suppose $EE'D'D$ to be the middle frustum of a parabolic spindle, V_2 its volume. Then, by the preceding Article,

$$V_2 = 2\pi k\left(\frac{8}{15}R^2 + \frac{4}{15}Rr + \frac{1}{5}r^2\right).$$

(3). Suppose $EE'D'D$ to be a double frustum of two equal paraboloids, V_3 its volume. The Prismoidal Formula, applied to each frustum, gives, on reduction,

$$V_3 = \pi k(R^2 + r^2).$$ See also Art. 91.

(4). Suppose $EE'D'D$ to be a double frustum of two equal cones, V_4 its volume. Then, by the Prismoidal Formula, or the formula for the volume of a frustum of a cone, Art. 87,

$$V_4 = \frac{2\pi k}{3}(R^2 + Rr + r^2).$$

For facility of comparison, we shall reduce these expressions to others, having a common factor $\frac{\pi l}{30}$ outside the brackets, l being the length of the cask, and equal to $2k$.

These expressions are

$$V_1 = \frac{\pi l}{30}(20R^2 + 10r^2),$$

$$V_2 = \frac{\pi l}{30}(16R^2 + 8Rr + 6r^2),$$

$$V_3 = \frac{\pi l}{30}(15R^2 + 15r^2),$$

$$V_4 = \frac{\pi l}{30}(10R^2 + 10Rr + 10r^2).$$

108. We shall now show that these four volumes are in descending order of magnitude, beginning with the first.

For
$$V_1 - V_2 = \frac{\pi l}{30}(4R^2 - 8Rr + 4r^2)$$
$$= \frac{4\pi l}{30}(R-r)^2,$$

a quantity necessarily positive; $\therefore V_1 > V_2$.

Again,
$$V_2 - V_3 = \frac{\pi l}{30}(R^2 + 8Rr - 9r^2)$$
$$= \frac{\pi l}{30}(R^2 - r^2 + 8Rr - 8r^2)$$
$$= \frac{\pi l}{30}(R + 9r)(R - r),$$

a positive quantity since R is $> r$; $\therefore V_2 > V_3$.

Lastly,
$$V_3 - V_4 = \frac{\pi l}{30}(5R^2 - 10Rr + 5r^2)$$
$$= \frac{5\pi l}{30}(R-r)^2; \therefore V_3 > V_4.$$

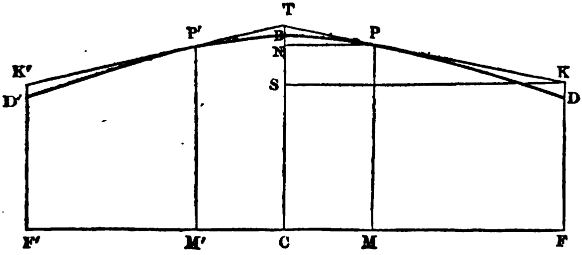
Hence V_1, V_2, V_3, V_4 are in descending order of magnitude.

By comparing the values of $V_1 - V_2$, $V_2 - V_3$, and $V_3 - V_4$, we see that the differences $V_1 - V_2$ and $V_3 - V_4$ are to each other as 4 to 5; and that they are very much smaller than the difference $V_2 - V_3$, since in all cases the difference between R and r is small compared with their sum.

Hence the *first two* varieties and the *last two* are very near to each other, but there is a much wider difference between the *second* and *third*.

109. In this Article we shall investigate the volume of a *compound* solid, the form of which was suggested by a careful examination of the actual construction of a number of casks, from which it appeared that the *curvature* of a cask, in the direction of its length, was confined to the middle part for about *one-third* of the whole length. The central part of the compound solid, thus suggested, consists of the middle frustum of a parabolic spindle; the other parts being conic frusta, equal to each other in all respects.

The solid will be one of revolution, and may be generated as follows:—



Let DBD' be an arc of a parabola, B its vertex, BC its axis, and FF' a portion of a chord perpendicular to the axis, and bisected in it.

Take $CM' = CM = \frac{1}{3} CF$; and through M, M' draw MP, $M'P'$ parallel to BC; and through P, P' draw the tangents TPK, $TP'K'$ meeting FD and $F'D'$ produced, in K and K'.

The solid generated by the revolution of the figure $FKPBP'K'F'$ round FF' will be the *compound* solid whose volume we proceed to find. The central portion will evidently be the middle frustum of a parabolic spindle, and the two ends equal conic frusta.

Let V be the volume of the central parabolic portion generated by the revolution of PP', v the volume of each conic frustum, generated by PK or $P'K'$, and V' the whole volume;

$$\therefore\ V' = V + 2v.$$

Let $CB = R$, $FK = r$, $MP = \rho$, $CF = k$; $\therefore\ CM = \frac{k}{3}$.

Then $$V = \frac{2\pi k}{3}\left(\frac{8}{15}R^2 + \frac{4}{15}R\rho + \frac{1}{5}\rho^2\right), \quad \text{Art. 106.}$$

$$\therefore\ V = \frac{2\pi k}{45}(8R^2 + 4R\rho + 3\rho^2);$$

also $$v = \frac{2\pi k}{9}(\rho^2 + \rho r + r^2). \quad \text{Art. 87.}$$

Substituting these values of V and v in the equation $V' = V + 2v$, we find

$$V' = \frac{2\pi k}{45}(8R^2 + 4R\rho + 13\rho^2 + 10\rho r + 10r^2);$$

and from this expression ρ must be eliminated because it is unknown.

Draw PN, KS parallel to CF, then the triangles PNT, KST are similar;

$$\therefore \frac{ST}{SK}=\frac{NT}{NP}, \text{ or } \frac{ST}{NT}=\frac{SK}{NP}=\frac{3}{1};$$

$$\therefore ST=3NT; \quad \therefore SN+NT=3NT, \text{ or } SN=2NT.$$

Now since BP is an arc of a parabola, BC its axis, and TP the tangent at P, we know that the subtangent NT is double of the abscissa BN;

$$\therefore 2NT=4BN; \quad \therefore SN=4BN.$$

But $$SN=CN-CS=MP-FK=\rho-r;$$

$$BN=CB-CN=CB-MP=R-\rho;$$

$$\therefore \rho-r=4(R-\rho);$$

$$\therefore 5\rho=4R+r.$$

Hence $$4R\rho=\frac{4}{5}(4R^2+Rr)=3.2R^2+0.8Rr;$$

$$13\rho^2=\frac{13}{25}(16R^2+8Rr+r^2)=8.32R^2+4.16Rr+0.52r^2;$$

$$10\rho r=2(4Rr+r^2)=8Rr+4r^2.$$

Substituting these values of $4R\rho$, $13\rho^2$ and $10\rho r$ in the expression for V', and reducing, we find

$$V'=\frac{2\pi k}{90}(39.04R^2+25.92Rr+25.04r^2).$$

In this formula we may omit the small quantity (.04) in the coefficients of R^2 and r^2, and add the double of it to the coefficient of Rr without making any appreciable difference in the approximation; and the *sum* of the numerical coefficients of R^2, Rr, and r^2 will be unaltered. Thus we obtain the more convenient formula

$$V'=\frac{\pi l}{90}(39R^2+26Rr+25r^2),$$

l being the length of the cask.

This formula gives the closest approximation to the true capacity of a cask which can be obtained from the three dimensions R, r, and l. It has been tested by comparing the values computed by it with the measured contents of a number of casks, and found to give very accurate results. The value of V' differs very little from the average value of V_1, V_2, V_3 and V_4, or from that of V_2 and V_3.

The formula for V' was first given by Dr Hutton.

It should be observed that the first variety V_1 is that which is employed by the Excise in gauging casks; but that it is always in excess of the true contents.

110. Another method of gauging a cask, in which *four* dimensions are required, is as follows:—

Let r' be the radius midway between R and r; then, applying the Prismoidal Formula to one-half of the cask, and doubling the result, we obtain

$$6V = 2\pi k\,(R^2 + r^2 + 4r'^2);$$

$$\therefore\ V = \frac{\pi l}{6}\,(R^2 + r^2 + 4r'^2).$$

It is, however, frequently difficult to measure the diameter $2r'$ with sufficient accuracy.

111. To determine the greatest error which can be made by supposing the true volume of a cask to be, (1) between V_1 and V_4; (2) between V_2 and V_3.

Let V be the true volume.

Then the error made by taking V_1 as the true volume is the $\left(\frac{V_1 - V}{V}\right)^{\text{th}}$ part of the whole.

But $V_1 - V < V_1 - V_4$, and $V > V_4$; Art. 108.

$$\therefore \text{ for both reasons, } \frac{V_1 - V}{V} < \frac{V_1 - V_4}{V_4};$$

$$\text{and since } \quad \frac{V_1 - V_4}{V_4} = \frac{10R^2 - 10Rr}{10R^2 + 10Rr + 10r^2}; \text{ Art. 107.}$$

$$\therefore \text{ the error is } < \frac{R(R-r)}{R^2 + Rr + r^2}.$$

Again, supposing the true volume to lie between V_2 and V_3,

$$\text{the error is} < \frac{V_2 - V_3}{V_3} < \frac{(R+9r)(R-r)}{15(R^2+r^2)}.$$

These errors will obviously be the least when $(R-r)$ is the least, i.e., when the head diameters differ the least from the bung diameter.

It is probable that if we take V_2 for the true volume, the *actual* error will be much smaller than that fraction of the whole which is represented by $\frac{V_2 - V_3}{V_3}$, and very much smaller if we take either V', or $\frac{1}{2}(V_2 + V_3)$ for the true volume.

112. **Examples of the preceding formulæ.**

Ex. 1. Suppose the interior length of a cask to be 30 inches, the bung diameter 24, and the head diameters 18 inches; find the capacity of the cask.

Here $R = 12,\ r = 9,\ l = 30;$

$$\begin{aligned}
\therefore\ V_1 &= \pi\,(2880 + 810) = \pi \times 3690; \\
V_2 &= \pi\,(2304 + 864 + 486) = \pi \times 3654; \\
V_3 &= \pi\,(2160 + 1215) = \pi \times 3375; \\
V_4 &= \pi\,(1440 + 1080 + 810) = \pi \times 3330; \\
V' &= \frac{\pi}{3}\,(5616 + 2808 + 2025) = \pi \times 3483.
\end{aligned}$$

It will be observed that V_1 and V_2 are near to each other, as also are V_3 and V_4, and that there is a much greater difference between V_2 and V_3; also, that V' lies between V_2 and V_3, and is nearly equal to the average of the other four values, or to that of the second and third, which are, respectively, $\pi \times 3512.25$ and $\pi \times 3514.5$.

Ex. 2. If the bung diameter be 24 inches, and the head diameters 20 inches; find the greatest error which can be made by taking either V_1 or V_4 for the capacity of the cask.

$$\text{The error is} < \frac{R(R-r)}{R^2+Rr+r^2};$$

$$\therefore \text{ error is} < \frac{12 \times 2}{144+120+100} < \frac{24}{364} < \frac{1}{15}.$$

Thus, in a cask of this form, containing 30 gallons, the error could not be so great as 2 gallons by the most inaccurate gauging, and would probably be less than one gallon.

If the interior length of the cask be 30 inches, we find

$$V_1 = 43.9 \text{ gallons}, \quad V_2 = 41{\cdot}2 \text{ gallons},$$

since 1 gallon $= 277.274$ cubic inches;

and these differ by 2.7 gallons, which is $< \frac{1}{15}$ of the *smaller* value.

Again, supposing the true value to lie between V_2 and V_3, the greatest error possible by taking either of these for the capacity of the cask is

$$< \frac{(R+9r)(R-r)}{15(R^2+r^2)};$$

$$\therefore \text{ error is} < \frac{(12+90) \times 2}{15\ (144+100)}$$

$$< \frac{102}{15 \quad 122} < \frac{51}{915} < \frac{1}{18}, \text{ very nearly.}$$

Hence, in this case, the error would be less than $\frac{1}{18}$th of the whole, and probably less than $\frac{1}{36}$th.

113. In the preceding Articles we have used R and r as most convenient in the investigations; but, in practice, the diameters D and d are measured, and not the radii R and r.

We have only to substitute $\frac{D}{2}$ for R and $\frac{d}{2}$ for r in the formulæ of Arts. 107, 109, when they become, after reduction,

$$V_1 = \frac{\pi l}{12}(2D^2 + d^2),$$

$$V_2 = \frac{\pi l}{60}(8D^2 + 4Dd + 3d^2),$$

$$V_3 = \frac{\pi l}{8}(D^2 + d^2),$$

$$V_4 = \frac{\pi l}{12}(D^2 + Dd + d^2),$$

$$V' = \frac{\pi l}{360}(39D^2 + 26Dd + 25d^2);$$

and these are the formulæ to be used in practice.

For their ready application to find the number of *gallons* which a cask will contain, a slight modification is necessary. The dimensions of a cask are always given in inches; therefore the unit of volume will be a cubic inch. Now, since there are 277.274 cubic inches in one imperial gallon, we must divide the formulæ by this number to obtain the capacity of the cask in imperial gallons.

Thus $V_1 = \frac{\pi}{277.274} \times \frac{l}{12}(2D^2 + d^2)$ imp. galls.,

or, since $\pi = 3.14159$, $\therefore \frac{\pi}{277.274} = .011330.$

Hence, if we substitute this last number for π in each of the above formulæ, they will give the capacity of the cask in *imperial gallons* instead of, as before, in *cubic inches.*

Thus the value of V' in Ex. 1, which is there expressed in cubic inches, becomes

$$V' = 3483 \times \cdot 01133 = 39\cdot 46 \text{ imperial gallons.}$$

CHAPTER XI.

ON THE METHODS OF APPROXIMATING TO THE AREAS OF FIGURES BOUNDED BY FREE CURVES, AND THE VOLUMES OF SOLIDS BOUNDED BY FREE SURFACES.

114. IN Chap. VII. we have shown how to determine the areas of plane curvilinear figures when the bounding curves are described according to defined laws; and we have determined the areas of some of the more important curves, such as the circle, ellipse, parabola, and others.

In Chap. VIII. we have investigated the volumes of certain solids, bounded by curve surfaces described according to known laws.

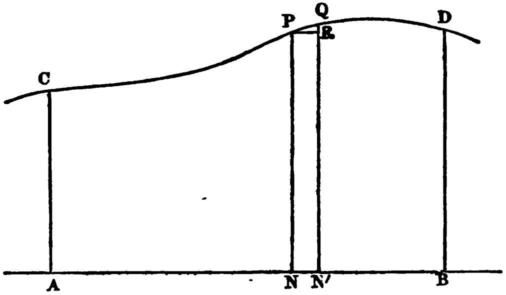
In this Chapter we shall show how to determine, to various degrees of approximation, the areas of curvilinear figures when the curvilinear boundaries are supposed to be traced with a free hand and *not* according to any known law; and we shall afterwards extend the formulæ to determine, approximately, the volumes of solids bounded by free surfaces.

We have already given two formulæ for approximating to the areas of curvilinear figures in Chap. III. We shall here complete the subject, adopting a method of investigation applicable to all cases.

The method briefly stated is this:—We determine a curve, *described according to a defined law,* and coinciding as closely as possible with the given curve; and we find the area of the figure bounded by *this* curve, instead of the area of the figure bounded by the *given* curve.

115. *General Explanation.*

Let CD be a portion of the boundary of a curvilinear figure, AB any convenient line of reference, AC and BD perpendicular to AB.



Let $AN = x$, $NP = y$, be the coordinates of any point P in the curve between C and D.

And suppose that

$$y = A + Bx + Cx^2 + \dots + Kx^n,$$

where A, B, C, ... K are constants.

Then, if this relation between x and y be true for *all* points in the curve between C and D, we can find the area of the curvilinear figure $ABDC$, accurately, by taking the limit of the sum of all the inscribed parallelograms.

But if this relation between x and y be true only for a *limited number* of points in the curve CD, then we can only find the area of a curvilinear figure very nearly coinciding with the given curvilinear figure, and the more nearly the greater the number of points which the two curves have in common.

Now the $(n+1)$ constants A, B, C, ... K may be determined so as to satisfy $(n+1)$ conditions.

Let these conditions be that the two curves shall have $(n+1)$ equidistant ordinates in common, of which AC and BD are, respectively, the first and last.

Divide AB into n equal parts, and let each of these parts $=h$, so that $AB=nh$.

And let $y_1, y_2, y_3, \ldots y_{n+1}$ be the values of the $(n+1)$ equidistant ordinates measured from the line AB to the *given* curve.

Then we have, by the conditions respecting them,

$$\begin{aligned}
y_1 &= A, \\
y_2 &= A + Bh + Ch^2 + \ldots + Kh^n, \\
y_3 &= A + 2Bh + 2^2Ch^2 + \ldots + 2^nKh^n, \\
y_4 &= A + 3Bh + 3^2Ch^2 + \ldots + 3^nKh^n, \\
\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
y_{n+1} &= A + nBh + n^2Ch^2 + \ldots + n^nKh^n.
\end{aligned}$$

From these $(n+1)$ equations the $(n+1)$ constants $A, B, C, \ldots K$ can be determined as functions of the $(n+1)$ *measured* ordinates $y_1, y_2, y_3, \ldots y_{n+1}$.

If this be done, the equation

$$y = A + Bx + Cx^2 + \ldots + Kx^n$$

will be the equation to a curve having $(n+1)$ points in common with the given curve, and its area will be an approximation to the area of the given curvilinear figure, and the more nearly the greater n is.

The actual determination of these constants is not necessary in any case. We shall suppose them determined, and find an expression for the area of the curve in terms of $A, B, C, \ldots K$, and afterwards eliminate these constants to obtain a convenient formula for the area in terms of the measured ordinates y_1, y_2, &c.

116. *To find an expression for the area of a curvilinear figure, the curvilinear boundary of which has* (n + 1) *points in common with a given curve.*

For this purpose suppose AB divided into m equal parts, and let AN be the first p of these parts measured from A, and NN' one of them; $\therefore NN' = \frac{nh}{m}$, and $AN = p\,\frac{nh}{m}$; AB being equal to nh as before.

Complete the parallelogram NR.

Then the area of this parallelogram $= NP \times NN'$.

But $NP = y = A + Bx + Cx^2 + \ldots + Kx^n$

$$= A + Bp\,\frac{nh}{m} + Cp^2\,\frac{n^2h^2}{m^2} + \ldots + Kp^n\,\frac{n^nh^n}{m^n};$$

$\therefore$ area of parallelogram NR

$$= nh\left(\frac{A}{m} + Bnh\,\frac{p}{m^2} + Cn^2h^2\,\frac{p^2}{m^3} + \ldots + Kn^nh^n\,\frac{p^n}{m^{n+1}}\right).$$

Hence the area of the curve will be found by giving to p the successive values 1, 2, 3,...m, and taking the limits of the sums of the resulting series when m is indefinitely increased;

$$\therefore \text{ area} = nh\left(A + \frac{B}{2}\,nh + \frac{C}{3}\,n^2h^2 + \ldots + \frac{K}{n+1}\,n^nh^n\right).$$

117. From this general expression for the area of the *assumed* curve, which has $(n+1)$ equidistant ordinates in common with the *given* curve, we shall deduce certain rules for approximating to the area of any curvilinear figure, according to the degree of approximation required.

(1). First approximation; $n = 1$.

Therefore the equation becomes

$$y = A + Bx,$$

and $$\text{area} = h\left(A + \frac{B}{2}\,h\right) = \frac{h}{2}(2A + Bh).$$

Also $$y_1 = A, \quad y_2 = A + Bh;$$

$$\therefore y_1 + y_2 = 2A + Bh;$$

$$\therefore \text{area} = \frac{h}{2}(y_1 + y_2).$$

This is the first method of approximation given in Chap. III., and is equivalent, as there assumed, to supposing CD a straight line instead of a curve, since the equation $y = A + Bx$ represents a straight line.

(2). Second approximation; $n = 2$.

Therefore the equation becomes

$$y = A + Bx + Cx^2,$$

and
$$\text{area} = 2h\left(A + Bh + \frac{4}{3}Ch^2\right)$$

$$= \frac{h}{3}(6A + 6Bh + 8Ch^2).$$

Also
$$y_1 = A,$$
$$y_2 = A + Bh + Ch^2,$$
$$y_3 = A + 2Bh + 4Ch^2,$$

$$\therefore y_1 + y_3 + 4y_2 = 6A + 6Bh + 8Ch^2.$$

Hence
$$\text{area} = \frac{h}{3}(y_1 + y_3 + 4y_2).$$

This is the second method of approximation given in Chapter III., and generally known as "Simson's Rule." It is equivalent, as there assumed, to supposing the curve an arc of a parabola having its axis perpendicular to AB, since the ordinate of any parabola in this position can be expressed in the form $A + Bx + Cx^2$. See Ex. 1, Art. 121.

(3). Third approximation; $n = 3$.

Therefore the equation becomes

$$y = A + Bx + Cx^2 + Dx^3,$$

and
$$\text{area} = 3h\left(A + \frac{3}{2}Bh + 3Ch^2 + \frac{27}{4}Dh^3\right)$$

$$= \frac{3}{8}h\{8A + 12Bh + 24Ch^2 + 54Dh^3\}.$$

Also
$$y_1 = A,$$
$$y_2 = A + Bh + Ch^2 + Dh^3,$$
$$y_3 = A + 2Bh + 4Ch^2 + 8Dh^3,$$
$$y_4 = A + 3Bh + 9Ch^2 + 27Dh^3;$$

$$\therefore\ y_1 + 3(y_2 + y_3) + y_4 = 8A + 12Bh + 24Ch^2 + 54Dh^3.$$

Hence $$\text{area} = \frac{3}{8} h \{y_1 + 3(y_2 + y_3) + y_4\}.$$

The fourth and fifth approximations, in which $n = 4$ and 5 respectively, do not lead to formulæ easy of application like the preceding.

We shall therefore pass them over, and consider, lastly, the sixth approximation, for which the equation is

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6.$$

This leads to the following remarkable formula for the approximate area, viz.

$$\text{area} = \frac{3}{10} h \{y_1 + y_3 + y_5 + y_7 + 5(y_2 + y_4 + y_6) + y_4\}.$$

This formula gives the closest approximation to the curvilinear area which can be obtained by any simple rule. It was first given by the late Mr Weddle in the *Mathematical Journal*, Vol. IX. p. 79, and has since been published in Professor Boole's *Finite Differences*.

The Student may prove the formula by following the same method as we have employed in the previous approximations. This method, however, involves considerable labour in this case. In the following Article we shall prove this important formula by a somewhat different method, much less laborious, though, perhaps, not quite as simple as the preceding.

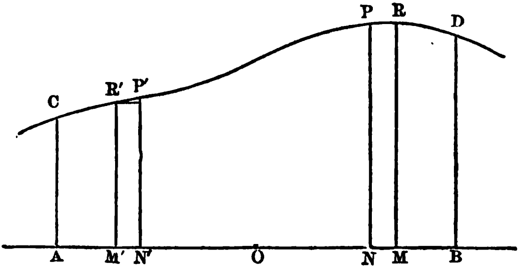
118. *Proof of Weddle's Rule.*

Let *ABDC* be any curvilinear figure as before. Bisect *AB* in *O*. Then, since in this case $n = 6$, $AB = 6h$, and therefore $OB = 3h$.

Divide OB into m equal parts; let ON be the first p of these, and NM one of them; therefore $NM=\frac{3h}{m}$, and $ON=p\frac{3h}{m}$.

Make $ON'=ON$, and $N'M'=NM$, and construct the parallelograms NR and $N'R'$ as in the figure.

Let O be the origin of coordinates; $ON=x$, $NP=y$, the coordinates of P; $ON'=x'$, $N'P'=y'$, those of P'.



$$\text{Therefore area of } NR=NP\times NM=y\frac{3h}{m},$$

$$\text{and area of } N'R'=N'P'\times N'M'=y'\frac{3h}{m};$$

$$\therefore \text{ sum of areas of } NR \text{ and } N'R'=\frac{3h}{m}(y+y').$$

But $y=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6$;

and $\quad y'=A-Bx+Cx^2-Dx^3+Ex^4-Fx^5+Gx^6$,

since, $x'=-x$;

$$\therefore y+y'=2(A+Cx^2+Ex^4+Gx^6);$$

Hence, if A_1, A_2 be the areas of NR and $N'R'$,

$$A_1 + A_2 = \frac{6h}{m}(A + Cx^2 + Ex^4 + Gx^6),$$

$$\therefore\ A_1 + A_2 = \frac{6h}{m}\left\{A + 3^2Cp^2\frac{h^2}{m^2} + 3^4Ep^4\frac{h^4}{m^4} + 3^6Gp^6\frac{h^6}{m^6}\right\},$$

putting for x its value $p\,\dfrac{3h}{m}$.

Hence the area of the whole curvilinear figure will be found by giving to p the successive values 1, 2, 3, ... m, and taking the limits of the resulting expressions;

$$\therefore\ \text{area} = 6h\left\{A + 3^2C\frac{h^2}{3} + 3^4E\frac{h^4}{5} + 3^6G\frac{h^6}{7}\right\},$$

or

$$\text{area} = 6h\left\{A + 3Ch^2 + \frac{81}{5}Eh^4 + \frac{729}{7}Gh^6\right\}.$$

The expression within the brackets contains only *even* powers of h; and, if we write down the values of the seven equidistant ordinates

$$y_1,\ y_2,\ y_3\ \ldots\ldots\ y_7,$$

it will be obvious that the only way in which we can combine them so as to avoid *odd* powers of h, is by taking them in pairs equidistant from the middle ordinate y_4.

Now

$$y_1 + y_7 = 2(A + 9Ch^2 + 81Eh^4 + 729Gh^6);$$

$$y_3 + y_5 = 2(A + Ch^2 + Eh^4 + Gh^6);$$

$$5(y_2 + y_6) = 10(A + 4Ch^2 + 16Eh^4 + 64Gh^6);$$

$$6y_4 = 6A.$$

$$\therefore\ y_1 + y_3 + y_5 + y_7 + 5(y_2 + y_4 + y_6) + y_4$$

$$= 20A + 60Ch^2 + 324Eh^4 + 2100Gh^6;$$

$$\therefore \frac{3}{10}h\{y_1 + y_3 + y_5 + y_7 + 5(y_2 + y_4 + y_6) + y_4\}$$

$$= 6h\left\{A + 3Ch^2 + \frac{81}{5}Eh^4 + \frac{735}{7}Gh^6\right\}.$$

If we compare this expression with that obtained above for the area, we see that they only differ *in their last terms*, and by the quantity

$$\frac{3^6}{7} Gh^7.$$

Therefore the value given by the formula is in excess of the area of the curve which we have taken as an approximation to the given curve, by the quantity

$$\frac{3^6}{7} Gh^7.$$

Therefore the error *in the last term*

$$= \frac{\frac{3^6}{7} Gh^7}{\frac{6 \times 729}{7} Gh^7}$$

$$= \frac{6}{729} = \frac{1}{121}, \text{ very nearly, or is } < \cdot 008,$$

or the error is less than $\frac{8}{1000}$ of the last term.

Hence the error in the whole quantity will be *very much smaller* than this, since the most important terms in the expression for y are the earlier terms.

We may therefore neglect this extremely small difference which will, probably in all cases, be very much less than the difference between the *true* area, and the approximate area given by the formula.

119. We shall now generalize the formulæ obtained in the last two Articles, so as to render them applicable to any curvilinear area, whatever may be the number of the ordinates which it may be desirable to measure.

Suppose there are $(n+1)$ equidistant ordinates

$$y_1, y_2, y_3, \ldots\ldots y_{n+1},$$

y_1 and y_{n+1} being the extreme or bounding ordinates; and let h be their common distance.

For the first approximation, n may be any number, odd or even; for the second approximation, or "Simson's Rule," n must be a multiple of 2; for the third n must be a multiple of 3; and for the sixth, or "Weddle's Rule," n must be a multiple of 6. Hence a multiple of 6 will suffice for all these approximations, and must be employed whenever it is desired to compare their results.

(1). Now, in the first approximation,

$$\text{Area between } y_1 \text{ and } y_2 = \frac{h}{2}(y_1 + y_2),$$

$$\text{..................} \; y_2 \text{.....} y_3 = \frac{h}{2}(y_2 + y_3),$$

$$\text{..................} \; y_3 \text{.....} y_4 = \frac{h}{2}(y_3 + y_4),$$

and so on;

$$\therefore \text{ Area between } y_1 \text{ and } y_4$$
$$= \frac{h}{2}\{y_1 + y_4 + 2(y_2 + y_3)\}.$$

From this we have the General Formula,—

$$\text{Area} = \frac{h}{2}\{\Sigma . y \,(\text{extreme}) + 2\,\Sigma . y \,(\text{intermediate})\},$$

"$\Sigma . y$ (intermediate)" standing for "the sum of all the ordinates which are intermediate to the first and last."

(2). In the second approximation,

$$\text{Area between } y_1 \text{ and } y_3 = \frac{h}{3}(y_1 + y_3 + 4y_2),$$

$$\text{..................} \; y_3 \text{......} y_5 = \frac{h}{3}(y_3 + y_5 + 4y_4),$$

$$\text{..................} \; y_5 \text{......} y_7 = \frac{h}{3}(y_5 + y_7 + 4y_6),$$

and so on;

$$\therefore \text{ Area between } y_1 \text{ and } y_7$$
$$= \frac{h}{3}\{y_1 + y_7 + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6)\}.$$

From this we have the General Formula,—

$$\text{Area}=\frac{h}{3}\{\Sigma\,.\,y(\text{extreme})+2\Sigma\,.\,y(\text{odd})+4\,\Sigma\,.\,y(\text{even})\}.$$

(3). In the third approximation,

$$\text{Area between } y_1 \text{ and } y_4 = \frac{3}{8}\,h\,\{y_1+y_4+3\,(y_2+y_3)\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ y_4 \ldots\ldots\ y_7 = \frac{3}{8}\,h\,\{y_4+y_7+3\,(y_5+y_6)\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ y_7 \ldots\ldots\ y_{10} = \frac{3}{8}\,h\,\{y_7+y_{10}+3\,(y_8+y_9)\},$$

and so on;

∴ Area between y_1 and y_{10}

$$=\frac{3}{8}\,h\,\{y_1+y_{10}+2\,(y_4+y_7)+3\,(y_2+y_3+y_5+y_6+y_8+y_9)\}.$$

Hence we have the General Formula,—

$$\text{Area}=\frac{3}{8}\,h\,\{\Sigma\,.\,y(\text{extreme})+2\Sigma\,.\,y(\text{every third})$$
$$+3\Sigma\,.\,y(\text{remaining})\}.$$

(4). In the sixth approximation,

$$\text{Area between } y_1 \text{ and } y_7 = \frac{3}{10}\,h\,\{y_1+y_3+y_5+y_7$$
$$+5\,(y_2+y_4+y_6)+y_4\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ y_7 \ldots\ldots\ y_{13} = \frac{3}{10}\,h\,\{y_7+y_9+y_{11}+y_{13}$$
$$+5\,(y_8+y_{10}+y_{12})+y_{10}\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ y_{13} \ldots\ldots\ y_{19} = \frac{3}{10}\,h\,\{y_{13}+y_{15}+y_{17}+y_{19}$$
$$+5\,(y_{14}+y_{16}+y_{18})+y_{16}\},$$

and so on;

∴ Area between y_1 and y_{19}

$$=\frac{3}{10}h\{y_1+y_3+y_5+y_7+y_9+y_{11}+y_{13}+y_{15}+y_{17}+y_{19}$$
$$+5(y_2+y_4+y_6+y_8+y_{10}+y_{12}+y_{14}+y_{16}+y_{18})$$
$$+y_4+y_7+y_{10}+y_{13}+y_{16}\}.$$

From this we deduce the General Formula,—

$$\text{Area}=\frac{3}{10}h\{\Sigma.y\,(\text{odd})+5\,\Sigma.y\,(\text{even})+\Sigma.y\,(\text{every third})\}.$$

120. The four General Formulæ which have been deduced in the last Article are as follows:—

(1) $$\text{Area}=\frac{h}{2}\{\Sigma.y\,(\text{extreme})+2\,\Sigma.y\,(\text{intermediate})\}.$$

(2) $$\text{Area}=\frac{h}{3}\{\Sigma.y\,(\text{extreme})+2\Sigma.y\,(\text{odd})+4\Sigma.y\,(\text{even})\}.$$

(3) $$\text{Area}=\frac{3h}{8}\{\Sigma.y\,(\text{extreme})+2\Sigma.y\,(\text{every third})+3\Sigma.y\,(\text{remaining})\}.$$

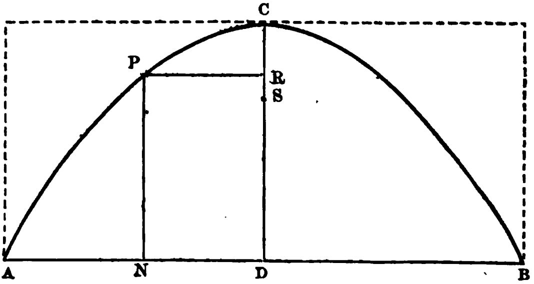
(4) $$\text{Area}=\frac{3h}{10}\{\Sigma.y\,(\text{odd})+5\Sigma.y\,(\text{even})+\Sigma.y\,(\text{every third})\}.$$

It is worthy of observation that in no one of these four formulæ are the *extreme ordinates* taken *twice;* this will remove all possible doubts as to the meaning of the expressions "Σ . *y* (odd)," "Σ . *y* (even)," "Σ . *y* (every third)", &c., from all of which the extreme ordinates are to be excluded;—excepting only the "Σ . *y* (odd)" in (4), in which the *extreme ordinates* have not been *previously* taken.

In applying these formulæ to determine, approximately, the areas of curvilinear figures, it is desirable to avoid great differences between consecutive ordinates. Where these differences would necessarily be considerable, a greater number of ordinates should be taken to determine the area of the curve.

121. In this Article we shall give two Examples in illustration of the formulæ proved in Arts. 117, 118. For these formulæ, see Art. 120.

Ex. 1. To find the area of a segment of a parabola cut off by a chord perpendicular to the axis.



Let *ACB* be a segment of a parabola, *C* its vertex, *S* its focus, and *AB* a chord perpendicular to the axis *CD*.

Take any point *P* in the curve; draw *PN* parallel to *CD*, and *PR* perpendicular to it.

Let $AN=x,\ NP=y,\ AD=h,\ CD=k,$

and let $a=4CS.$

By a well-known property of the parabola,

$$PR^2=4CS.CR.$$

But $$PR=ND=AD-AN=h-x,$$

$$CR=CD-DR=CD-PN=k-y$$

therefore, by substitution,

$$(h-x)^2=a(k-y),$$

$$\therefore\ h^2-2hx+x^2=ak-ay:$$

but $$h^2=ak;\ \therefore\ ay=2hx-x^2;$$

$$\therefore\ y=\frac{2h}{a}x-\frac{x^2}{a}.$$

This equation is of the form

$$y = A + Bx + Cx^2;$$

in which A is zero, because the origin of coordinates is a point.

Hence Simson's Rule will determine the area of this curve *accurately*;—and this is what we should expect when we bear in mind that Simson's Rule is founded on the principle of substituting a portion of a parabola (with its axis in the position here indicated) for any other curve between two near points.

By this rule,

$$\text{Area} = \frac{h}{3}(y_1 + y_3 + 4y_2).$$

Here $\qquad y_1 = 0, \quad y_3 = 0, \quad y_2 = k, \quad h = h;$

$$\therefore \text{ Area} = \frac{h}{3} 4k = \frac{4}{3} hk = \frac{2}{3}(2h)k = \frac{2}{3} AB \times CD;$$

or area = two-thirds of the circumscribed parallelogram.

This agrees with the result of Art. 76.

Ex. 2. To find, approximately, the area of a portion of a circle contained between two ordinates, equidistant from the centre, and distant from each other by the radius of the circle.

We shall apply each of the formulæ of Art. 120 to this problem, for the sake of comparing the approximations obtained by them.

Let AB, CD be the two ordinates, O the centre of the circle, and a its radius; then $AC = a$.

Divide AC into six equal parts, and draw the ordinates as in the figure; and let $y_1, y_2, y_3, y_4, y_5, y_6, y_7$ be the lengths of the seven equidistant ordinates, and h their common distance;

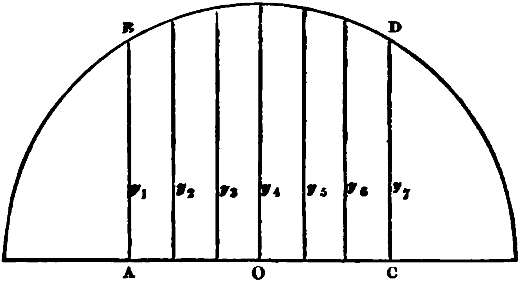
hen $\qquad 6h = a, \ y_1 = y_7, \ y_2 = y_6, \ y_3 = y_5, \ y_4 = a.$

Now $\qquad y_5 = \sqrt{a^2 - h^2} = \sqrt{a^2 - \frac{a^2}{6^2}} = \frac{a}{6}\sqrt{35}.$

Similarly, $y_6 = \frac{a}{6}\sqrt{32}$; $y_7 = \frac{a}{6}\sqrt{27}$;

$$\therefore\ y_1 = y_7 = \frac{a}{6}\sqrt{27}; \qquad y_2 = y_6 = \frac{a}{6}\sqrt{32};$$

$$y_3 = y_5 = \frac{a}{6}\sqrt{35}; \qquad y_4 = a.$$



(1). **For the first approximation,**

$$\text{Area} = \frac{h}{2}\{\Sigma.y\ (\text{extreme}) + 2\Sigma.y\ (\text{intermediate})\}$$

Here $\Sigma.y\ (\text{extreme}) = y_1 + y_7 = 2y_7 = \frac{a}{3}\sqrt{27}$;

$$2\Sigma.y\ (\text{intermediate}) = 2(y_2 + y_3 + y_4 + y_5 + y_6)$$

$$= 2y_4 + 4y_5 + 4y_6$$

$$= 2a + \frac{2a}{3}\sqrt{35} + \frac{2a}{3}\sqrt{32};$$

$$\therefore\ \text{Area} = \left(\frac{a}{3}\sqrt{27} + 2a + \frac{2a}{3}\sqrt{35} + \frac{2a}{3}\sqrt{32}\right)$$

$$= \frac{a^2}{36}(\sqrt{27} + 6 + 2\sqrt{35} + 2\sqrt{32})$$

$$= \frac{a^2}{36}(6 + 8\sqrt{2} + 3\sqrt{3} + 2\sqrt{35}).$$

Performing the arithmetical operations, we find

$$\text{Area} = a^2 \times .953945.$$

(2). For the second approximation, or "Simson's Rule,"

$$\text{Area} = \frac{h}{3}\{\Sigma . y\,(\text{extreme}) + 2\Sigma . y\,(\text{odd}) + 4\Sigma . y\,(\text{even})\}.$$

Here $\Sigma . y\,(\text{extreme}) = y_1 + y_7 = 2y_7 = \frac{a}{3}\sqrt{27};$

$$2\Sigma . y\,(\text{odd}) = 2(y_3 + y_5) = 4y_5 = \frac{2a}{3}\sqrt{35};$$

$$4\Sigma . y\,(\text{even}) = 4(y_2 + y_4 + y_6) = 4y_4 + 8y_6$$

$$= 4a + \frac{4a}{3}\sqrt{32};$$

$$\therefore \text{Area} = \frac{h}{3}\left(\frac{a}{3}\sqrt{27} + \frac{2a}{3}\sqrt{35} + 4a + \frac{4a}{3}\sqrt{32}\right)$$

$$= \frac{a^2}{54}(12 + 16\sqrt{2} + 3\sqrt{3} + 2\sqrt{35}).$$

Hence $\text{Area} = a^2 \times .956588.$

(3). For the third approximation,

$$\text{Area} = \frac{3h}{8}\{\Sigma . y\,(\text{extreme}) + 2\Sigma . y\,(\text{every third})$$

$$+ 3\Sigma . y\,(\text{remaining})\}.$$

Here $\Sigma . y\,(\text{extreme}) = y_1 + y_7 = 2y_7 = \frac{a}{3}\sqrt{27};$

$$2\Sigma . y\,(\text{every third}) = 2y_4 = 2a;$$

$$3\Sigma . y\,(\text{remaining}) = 3(y_2 + y_3 + y_5 + y_6)$$

$$= 6(y_5 + y_6) = a\sqrt{35} + a\sqrt{32}.$$

$$\therefore \text{Area} = \frac{3h}{8}\left(\frac{a}{3}\sqrt{27} + 2a + a\sqrt{35} + a\sqrt{32}\right)$$

$$= \frac{a^2}{16}(2 + 4\sqrt{2} + \sqrt{3} + \sqrt{35}).$$

Hence $\quad$ Area $= a^2 \times .956605.$

(4). For the sixth approximation, or "Weddle's Rule,"

$$\text{Area} = \frac{3h}{10}\{\Sigma . y\,(\text{odd}) + 5\Sigma . y\,(\text{even}) + \Sigma . y\,(\text{every third})\}.$$

Here $\Sigma . y\,(\text{odd}) = y_1 + y_3 + y_5 + y_7 = 2\,(y_5 + y_7)$

$$= \frac{a}{3}\sqrt{35} + \frac{a}{3}\sqrt{27};$$

$$5\Sigma . y\,(\text{even}) = 5\,(y_2 + y_4 + y_6) = 5\,y_4 + 10 y_6$$

$$= 5a + \frac{5a}{3}\sqrt{32};$$

$$\Sigma . y\,(\text{every third}) = y_1 = a;$$

$$\therefore \text{Area} = \frac{3h}{10}\left(\frac{a}{3}\sqrt{35} + \frac{a}{3}\sqrt{27} + 5a + \frac{5a}{3}\sqrt{32} + a\right)$$

$$= \frac{a^2}{60}(18 + 20\sqrt{2} + 3\sqrt{3} + \sqrt{35}).$$

Hence $\quad$ Area $= a^2 \times .956608.$

Now the area *ABDC* is manifestly the difference between a semicircle, and a segment whose height is half the radius.

From this we find

$$\text{Area} = a^2\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right);$$

and since $\pi = 3.1415927$; and $\sqrt{3} = 1.7320508$;

$$\therefore \text{Area} = a^2 \times .956612.$$

If we compare this value with the four approximate alues already found, viz.—

(1) Area $= a^2 \times .953945,$

(2) Area $= a^2 \times .956588,$

(3) Area $= a^2 \times .956605,$

(4) Area $= a^2 \times .956608,$

we see that the first is true to *two* places of decimals, the second to *four* places, and the third and fourth each to *five* places—the last being the nearest approximation.

122. The investigations and the results of the preceding Articles of this Chapter will also be applicable to the volumes of solids bounded by *free* surfaces; if, for the *lengths of the equidistant ordinates*, we substitute the *areas of the equidistant parallel sections.*

In this case we determine the volume of a solid which has a number of equidistant sections in common with the given solid, and the surface of which follows a known law; and the volume of this solid will be an approximation to the volume of the given solid, and the more nearly the greater the number of the sections which the two solids have in common.

Thus, in the second approximation, or "Simson's Rule," the expression for the area between two alternate ordinates is

$$\frac{h}{3}(y_1+y_3+4y_2);$$

and if A_1, A_2, A_3 be the areas of the equidistant parallel sections of a solid, this expression becomes, by substitution,

$$\frac{h}{3}(A_1+A_3+4A_2),$$

which is an approximate expression for the volume of the solid between the alternate sections A_1 and A_3.

Hence we have

$$6V=2h(A_1+A_3+4A_2);$$

and this is the "Prismoidal Formula," since A_1, A_3 are the areas of the ends, A_2 is the area of the mean section, and $2h$ is the length of the solid.

Again, by the general form of "Weddle's Rule," applied to volumes, we have

$$V=\frac{3}{10}h\{\Sigma . A\,(\text{odd})+5\Sigma . A\,(\text{even})+\Sigma . A\,(\text{every third})\},$$

a multiple of $6h$ being the length of the solid.

By means of this formula we may find the volume of any solid in which the areas of the transverse sections can be expressed by a rational integral algebraical function, of a degree not higher than the *fifth*, of its distance from either end; and, *very approximately*, when the expression is of the *sixth* degree; whereas in the case of the Prismoidal Formula, the degree must not be higher than the *third*.

Thus, for example, we may find the volume of the middle frustum of a parabolic spindle, already given in Art. 106. Referring to the figure of that Art. let GF be divided into six equal parts, each $=h$, and let

$$A_1, A_2, A_3, A_4, A_5, A_6 \text{ and } A_7$$

be the areas of the sections at the points of division, A_1 being that of the end EE'.

Let $CN=x$, $NP=y$ be the coordinates of any point P in the curve; $BC=R$, $DF=r$;

$$\therefore\ x^2=a(R-y), \qquad \therefore\ y=R-\frac{x^2}{a};$$

$$\therefore\ \pi y^2=\pi\left(R^2-2R\frac{x^2}{a}+\frac{x^4}{a^2}\right)$$

will be the area of the section at the distance x from C; and this being a rational integral function of x of the *fourth* degree, Weddle's Rule will determine the volume accurately.

Now $$A_1=A_7=\pi r^2;$$

$$A_2=A_6=\pi\left(R^2-2R\frac{4h^2}{a}+\frac{16h^4}{a^2}\right);$$

$$A_3=A_5=\pi\left(R^2-2R\frac{h^2}{a}+\frac{h^4}{a^2}\right);$$

and $$A_4=\pi R^2;$$

$$\therefore\ \Sigma . A\,(\text{odd}) = A_1 + A_3 + A_5 + A_7$$

$$= 2\pi r^2 + 2\pi \left(R^2 - 2R\,\frac{h^2}{a} + \frac{h^4}{a^2}\right);$$

$$5\Sigma . A\,(\text{even}) = 5\,(A_2 + A_4 + A_6)$$

$$= 5\pi \left(3R^2 - 16R\,\frac{h^2}{a} + 32\,\frac{h^4}{a^2}\right);$$

$$\Sigma . A\,(\text{every third}) = A_4 = \pi R^2.$$

$$\therefore\ \Sigma . A\,(\text{odd}) + 5\Sigma . A\,(\text{even}) + \Sigma . A\,(\text{every third})$$

$$= \pi \left(18R^2 + 2r^2 - 84R\,\frac{h^2}{a} + 162\,\frac{h^4}{a^2}\right);$$

but $(3h)^2 = a\,(R - r)$; $\therefore\ \dfrac{h^2}{a} = \dfrac{R - r}{9}$;

$$\therefore\ \Sigma . A\,(\text{odd}) + 5\Sigma . A\,(\text{even}) + \Sigma . A\,(\text{every third})$$

$$= \pi \left(18R^2 + 2r^2 - \frac{28}{3}\,R\,(R - r) + 2\,(R - r)^2\right);$$

$$= \frac{\pi}{3}\,(32R^2 + 16Rr + 12r^2)$$

$$= \frac{4}{3}\,\pi\,(8R^2 + 4Rr + 3r^2).$$

Hence, by the formula

$$V = \frac{3}{10}\,h\,\{\Sigma . A\,(\text{odd}) + 5\Sigma . A\,(\text{even}) + \Sigma . A\,(\text{every third})\},$$

$$V = \frac{3}{10}\,h\,\frac{4}{3}\,\pi\,(8R^2 + 4Rr + 3r^2)$$

$$= \frac{6\pi h}{15}\,(8R^2 + 4Rr + 3r^2).$$

This is identical with the expression for the middle frustum, obtained in Art. 106, since $6h = 2k$.

A COLLECTION OF EXAMPLES AND PROBLEMS.

Examples and Problems on Chapter I.

1. The base of the Great Pyramid of Egypt is a square of 764 feet in each side; find the number of acres of ground covered by it.

By Arts. 3 or 4,

Area = 764 × 764 = 583696 sq. feet,
= 64855 sq. yards 1 sq. foot,
= 2143 sq. poles 29 sq. yards 1 sq. foot,
= 53 roods 23 sq. poles 29 sq. yards 1 sq. foot,
= 13 acres 1 rood 23 sq. poles 29 sq. yards 1 sq. foot.

Hence the area covered by the base is rather more than $13\frac{1}{4}$ acres.

2. The sides of a triangular field are 32.13, 33.96, 48.39 chains; find the area of the field in acres.

By Art. 6, Case (3),

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)},$$

and $\therefore$ $2\log \text{area} = \log s + \log(s-a) + \log(s-b) + \log(s-c)$.

Here $2s = 32.13 + 33.96 + 48.39 = 114.48$;

$\therefore$ $s = 57.24$; $(s-a) = 25.11$; $(s-b) = 23.28$; $(s-c) = 8.85$.

$$\begin{aligned} \therefore \log s &= \log 57.24 = 1.7576996, \\ \log(s-a) &= \log 25.11 = 1.3998467, \\ \log(s-b) &= \log 23.28 = 1.3669830, \\ \log(s-c) &= \log 8.85 = 0.9469433; \end{aligned}$$

$$\therefore\ 2 \log \text{area} = 5.4714726;$$

$$\therefore\ \log \text{area} = 2.7357363.$$

By Tables $\log 544.17 = 2.7357346;$

$$\therefore\ \text{difference} = .0000017,$$

$$\text{also } D = .0000080;$$

$\therefore$ the last two digits are those in $\frac{1700}{80}$, or 21.

Hence $\text{area} = 544.1721$ sq. chains

$= 54$ acres, 4.1721 sq. chains.

This result is true within one square link: if it were only required to find the area true to one-tenth of a square chain, we should reject the last two digits and write the result, 54 acres, 4.2 sq. chains.

3. The diagonals of a quadrilateral enclosure are 17.21, and 24.32 chains, and the angle between them is $39^0\ 14'$; find the area of the enclosure, true to the hundredth part of an acre.

By Art. 7, Case (2), we have the formula

$$\log \text{area} = \log d_1 + \log d_2 + \log \sin \theta - \log 2;$$

$$\therefore\ \log \text{area} = \log d_1 + \log d_2 + \text{Log} \sin \theta - 10 - \log 2,$$

where Log $\sin\theta$ is the tabular logarithm of $\sin\theta$, or the true logarithm increased by 10.

Here
$$\begin{aligned} \log d_1 &= \log 17.21 = 1.2357809, \\ \log d_2 &= \log 24.32 = 1.3859636, \\ \text{Log} \sin \theta &= \text{Log} \sin 39^0\ 14' = 9.8010468 \\ -10 - \log 2 &= -10.3010300; \end{aligned}$$

$$\therefore\ \log \text{area} = 2.1217613.$$

By Tables $\log 132.36 = 2.1217568;$

$$\therefore\ \text{area} = 132.36 \text{ sq. chains} = 13.236 \text{ acres},$$

or area $= 13.24$ acres, true to the second decimal, or to the hundredth part of an acre.

4. Find the area of a regular octagon, each side of which is a unit of length.

For this purpose we require the formula of Art. 11, (3), which is

$$A = n\frac{a^2}{4}\cot\frac{180^0}{n};$$

and here $n=8$, $a=1$, and therefore

$$\frac{180^0}{n} = \frac{180^0}{8} = \frac{45^0}{2};$$

$$\therefore\ A = 2\cot\frac{45^0}{2}.$$

Now

$$\cot^2\frac{45^0}{2} = \frac{1+\cos 45^0}{1-\cos 45^0} = \frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = (\sqrt{2}+1)^2;$$

$$\therefore\ \cot\frac{45^0}{2} = 1+\sqrt{2} = 2.4142136\ldots$$

$$\therefore\ A = 2\times 2.4142136 = 4.8284272.$$

The same result may of course be obtained by logarithms from the formula

$$\log A = \log 2 + \text{Log}\cot 22^0\,30' - 10.$$

From the numerical value last found, the area of any other octagon may be obtained when we know the length of its side; for, if a be the length of the side, the area will be

$$(4.8284272)\,a^2,$$

by the formula quoted above, or since the areas of *similar* polygons are in the duplicate ratio of their homologous sides.

Thus, if the side of an octagon be 10 feet, its area will be 10^2, or 100 times that of an octagon whose side is one foot; therefore its area will be 482.84272 sq. feet.

This suggests the advantage of a table of the areas of the regular polygons, when the sides are units of length.

If the student calculate the areas of the first ten regular polygons when the side is a unit of length, he will easily construct the following table.

Number of sides.	Name of Polygon.	Area when side = 1.
3	Equilateral triangle	0.4330127
4	Square	1.0000000
5	Pentagon	1.7204774
6	Hexagon	2.5980762
7	Heptagon	3.6339124
8	Octagon	4.8284271
9	Nonagon	6.1818242
10	Decagon	7.6942088
11	Undecagon	9.3656399
12	Dodecagon	11.1961524

The Heptagon, Nonagon, and Undecagon are most conveniently calculated by means of the Logarithmic Tables; the rest may be easily computed as surds, requiring only $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$.

5. If the alternate sides of a regular polygon of an *even* number of sides be produced to form another regular polygon of half the number of sides; find its area in terms of the area of the given polygon.

Let A be the area of the given polygon of $2n$ sides,
A_1 .... new polygon of n sides.

The two polygons will evidently have the same inscribed circle: let r be its radius;

$$\therefore\ A = 2nr^2 \tan\frac{\pi}{2n},\quad A_1 = nr^2 \tan\frac{\pi}{n};\quad \text{Art. 11, (2)};$$

$$\therefore\ \frac{A_1}{A} = \frac{\tan\frac{\pi}{n}}{2\tan\frac{\pi}{2n}} = \frac{2\tan\frac{\pi}{2n}}{2\tan\frac{\pi}{2n}\left(1-\tan^2\frac{\pi}{2n}\right)};$$

$$\therefore\ \frac{A_1}{A} = \frac{1}{1-\tan^2\frac{\pi}{2n}} = \frac{\cos^2\frac{\pi}{2n}}{\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{2n}} = \frac{\cos^2\frac{\pi}{2n}}{\cos\frac{\pi}{n}}.$$

Hence
$$A_1 = \frac{\cos^2\frac{\pi}{2n}}{\cos\frac{\pi}{n}}A.$$

Thus, if $n=3$, $A_1=\dfrac{\cos^2\frac{\pi}{6}}{\cos\frac{\pi}{3}}A=\dfrac{\frac{3}{4}}{\frac{1}{2}}A=\dfrac{3}{2}A,$

or the equilateral triangle formed by producing the alternate sides of a regular hexagon is greater than the hexagon by one-half.

6. The area of the base of a frustum of a regular hexagonal pyramid is double of the area of the top; the perimeter of the top is 6 feet, and the slant height 9 feet; find the area of the whole surface.

By Art. 16, Cor.

$$\text{area of whole surface}=A_1+A_2+\frac{h}{2}(P+p).$$

Here $A_1=2A_2$; $\therefore$ $P^2=2p^2$, or $P=p\sqrt{2}$;

$$\therefore \text{ area}=3A_2+\frac{hp}{2}(1+\sqrt{2}).$$

Also $A_2=n\dfrac{a^2}{4}\cot\dfrac{180^0}{n}=\dfrac{6}{4}\cot 30^0=\dfrac{3}{2}\sqrt{3},$

and $h=9$, $p=6$;

$$\begin{aligned}\therefore \text{ area}&=\frac{9}{2}\sqrt{3}+27(1+\sqrt{2})\\&=\frac{9}{2}1.732+27\times 2.414\\&=72.98 \text{ sq. feet,}\\&=72 \text{ sq. ft. } 141.12 \text{ sq. inches.}\end{aligned}$$

7. Find the number of square yards in the floor of a rectangular room, the sides of which are 40 and 20 feet.

Ans. 88 sq. yds. 2 sq. ft.

8. The sides of a rectangular field are 12.25 and 8.50 chains; find its area. Ans. 10 ac. 1 rd. 26 sq. po.

9. The diagonal of a square court is 300 feet; find its area in square yards. Ans. 5,000 sq. yards.

10. A square field contains 31 acres 0 roods 10.25 square poles; find the length of a side, and of the diagonal.

Ans. side = 17.625 chains.

11. A rectangular grass-plot, the sides of which are as 2 : 3, cost £14. 8*s*. for turfing, at the rate of 4*d*. per square yard; find the lengths of its sides. Ans. 24 and 36 yards.

12. A gravel walk of uniform breadth is made round a rectangular grass-plot, the sides of which are 20 and 30 yards; find the breadth of the walk if its area be three-tenths of that of the grass-plot. Ans. 10 feet 2.4 inches.

13. A room 40 feet long, 22 wide, and 15 high, has four windows in one side, each occupying a space 7 feet wide and 13 feet high; two doors, each 4 feet 9 inches wide and 8 feet high; and two fire-places, each 5 feet 8 inches wide, and 4 feet 6 inches high: find the number of square yards of painting required for the walls, allowing 1 foot for the height of the cornice, and 15 inches for the skirting-board.

Ans. 1089 sq. yds. 1 sq. foot.

14. One side of a parallelogram is 300 feet, and the perpendicular from the opposite side is 190 feet; find the area in square yards. Ans. $6333.\dot{3}$.

15. A parallelogram may be divided into a number of equal parallelograms in the same manner as the rectangle on page 2 is divided into a number of equal rectangles, and the number of these parallelograms will manifestly be the product of the numbers of units of length in two adjacent sides; why, then, cannot these parallelograms be taken for units of area?

16. Two sides of a parallelogram are 10.62 and 15.35 chains, and the angle between them is 30^0; find the area.

Ans. 8.156 acres.

17. Two sides of a parallelogram are 20 and 30 feet, and the area is 450 square feet; find the angle between the sides. Ans. $\sin^{-1}\frac{3}{4}$, or between $48^0\ 35'$ and $48^0\ 36'$.

18. The diagonals of a field, the sides of which are all equal, are 20 and 30 chains; find its area. Ans. 30 acres.

19. The sides of a rhombus are 21 feet 3 inches, and the smaller angle is $53^0\ 20'$; find the area in square yards.

Ans. 40.2232.

20. The diagonals of a parallelogram are 25 and 30 feet, and its area is 325 square feet; show that the angle between the diagonals is 60^0 very nearly.

21. The base of a triangle is 7 feet, and the altitude 8 feet; find its area. Ans. 9 sq. yds. 1 sq. ft.

22. Two sides of a triangular field are 4 and 5 chains, and they include a right angle; find the area. Ans. 1 acre.

23. Two sides of a triangle are 25 and 21.25 chains, and the included angle 45°; find the area.

Ans. 18 ac. 3 rd. 5 sq. po.

24. Find the area of an equilateral triangle, each side of which is 21 feet. Ans. 191.1 sq. feet.

25. The perimeter of an equilateral triangle is 200 feet; find its area. Ans. 1924.5 sq. feet.

26. If p be the perpendicular of an equilateral triangle, show that its area is $\frac{p^2}{\sqrt{3}}$.

27. The sides of a triangular field are 20, 30, and 40 chains; find its area. Ans. 29 ac. 0 rd. 7 sq. po.

28. The sides of a triangular field are 49, 50.25, and 25.69 chains; find its area. Ans. 61 ac. 3 rd. 37 sq. po.

29. The base of an isosceles triangle is 40 feet, and the vertical angle 120°; find the area. Ans. 230.94 sq. feet.

30. The area of a triangle is 20.869 square feet, one angle is 45°, and one of the sides which contain this angle is 25 feet; find the other side. Ans. 21.25 feet.

31. One diagonal of a quadrilateral is 70 feet, and the sum of the perpendiculars upon it is 24 feet; find its area.

Ans. 840 sq. feet.

32. One diagonal of a quadrilateral, which lies outside the figure, is 70 feet, and the difference of the perpendiculars upon it is 16 feet; find the area. Ans. 560 sq. feet.

33. One diagonal of a quadrilateral is 321 feet, and the perpendiculars from the opposite angular points are 108 and 173 feet; find the area in square yards, (1) supposing the diagonal *within*, (2) supposing it *without* the figure.

Ans. (1) 90201 sq. feet, (2) 20865 sq. feet.

34. The diagonals of a quadrilateral are 30 and 40 chains, and they intersect at an angle of 45°; find the area.

Ans. 44 ac. 2 rd. 14 sq. po.

35. The diagonals of a quadrilateral field are 21.81 and 33.20 chains, and they include an angle of 72^0; find the area.
Ans. 34.441 acres.

36. The diagonals of a quadrilateral are 305 and 208, and the angle between them is 55^0; also, the perpendiculars on the longer diagonal are 79 and 91.4: calculate the area from each set of data.

37. One side of a quadrilateral field is 11.1 chains, and the perpendiculars from the opposite corners are 3.52 and 5.95 chains; the foot of one perpendicular is distant 1.1 chain from one end of the given side, and the foot of the other 3.65 chains from the other end: find the area. Ans. 4.286 acres.

38. The sides of a quadrilateral, taken in order, are 143, 95.3, 132, 187 feet, and the angles between the first two and the last two are 78^0 and $55^0\,15'$ respectively; find the area.
Ans. 16806 sq. feet.

39. If the four sides of a quadrilateral be given and *one* of the angles, show that the figure is determinate when it is known whether the *opposite* angle is greater or less than two right angles.

40. The sides of a quadrilateral field, taken in order, are 650, 1296, 1481 and 423 links; the angle between the first two is 125^0 and the opposite angle less than two right-angles: find the area of the field. Ans. 611228 sq. links.

41. Three sides of a quadrilateral field, taken in order, are 23.4, 19.75 and 20.5 chains; the angle between the second and third is 73^0, and the opposite angle $87^0\,30'$: find the area.
Ans. 34.315 acres.

42. The sides of a pentagon, taken in order, are 100, 130, 197, 133, and 94 feet, and the two diagonals measured from the intersection of the first and last sides are 209 and 193 feet; find the area of the figure. Ans. 3090.3 sq. yards.

43. The parallel sides of a trapezoid are 7.5 and 12.25 chains, and the perpendicular is 15.4 chains; find the area.
Ans. 15 ac. 33 sq. poles.

44. The parallel sides of a trapezoid are 55 and 77 feet, and the other sides are 25 and 31 feet; find the area.
Ans. 1590.6 sq. feet.

45. The parallel sides of a trapezoid are 39 and 56 yards, and the angles which the other two sides make with the longer side are 45^0 and 57^0; find the area. Ans. 489.58 sq. yards.

46. Bisect a trapezoid by a straight line passing through a point in the greater of the two parallel sides, and through one of the opposite angular points.

Ans. If x be the distance of the point from *one* extremity of the side, $x=\frac{a+b}{2}$, a and b being the lengths of the parallel sides.

47. Show that the problem to bisect a trapezoid by a straight line perpendicular to the parallel sides is indeterminate, if only the lengths of these sides and their distance be given.

48. A trapezoid, the parallel sides of which are as 3 : 4, is to be cut from a rectangular board 12 feet long and 2 feet wide; find the lengths of the parallel sides that the trapezoid may be one-third of the board.

Ans. 3 ft. $5\frac{1}{7}$ in., and 4 ft. $6\frac{6}{7}$ in.

49. The sides of a quadrilateral, taken in order, are 75, 55, 60 and 40 inches, and the angle between the first two sides is $74^0\ 40'\ 15''$; show that the figure may be inscribed in a circle, and find its area. Ans. Area = 21.85 sq. feet.

50. The sides of a quadrilateral inscribed in a circle are 6, 5.5, 7.5 and 4 yards; find the area, and the angles between the first and last, and the second and third sides.

Ans. Area = 31.46 sq. yds.; angles = $74^0\ 40'\ 15''$ and $105^0\ 19'\ 45''$.

51. A quadrilateral in a circle is bisected by one of its diagonals, and trisected by the other; if a, c be two adjacent sides which contain an obtuse angle, and are subtended by the trisecting diagonal, show that the area

$$=\frac{1}{4}\sqrt{\{34a^2c^2-(a^4+c^4)\}}.$$

52. If R be the radius of the circle which circumscribes a quadrilateral, a, b, c, d the four sides, and $2s$ the perimeter; show that

$$R=\frac{1}{4}\sqrt{\left(\frac{(ab+cd)\ (ac+bd)\ (ad+bc)}{(s-a)\ (s-b)\ (s-c)\ (s-d)}\right)}.$$

53. Find the values of R and r, and the areas of the following regular polygons:—(1) a pentagon; (2) a hexagon; (3) an octagon; (4) a decagon—the length of one side being 25 feet.

Ans.

R=(1) 21.27; (2)·25; (3) 32.67; (4) 40.41 ft.
r=(1) 17.21; (2) 21.65; (3) 30.18; (4) 38.47 ft.
A=(1) 1075.2; (2) 1623.8; (3) 3017.8; (4) 4808.9 sq. ft.

54. Find the side of an octagon which has an area of 1000 sq. yards. Ans. 14 yds. 1 ft. 2.4 inches.

55. The perimeter of a quindecagon is 135 feet; find its area, and the radii of the inscribed and circumscribed circles.
Ans. Area = 1429 sq. feet.

56. Each diagonal of a pentagon is 10 feet; find its area.
Ans. 65.72 sq. feet.

57. A pentagon and a decagon have one side common; find the area of the space between them in terms of one side.
Ans. If a be the side, area $= 5.97382\,a^2$.

58. The longest diagonal of a nonagon is 15 feet; find its area. Ans. 168.54 sq. feet.

59. The area of a regular polygon of 25 sides is 315.823 sq. feet; find the radius of the inscribed circle, and show that if the radius be increased by 1 foot, the area of the polygon will be increased by 66.323 sq. feet.

60. The radius of a circle being 30 feet, find the side of an inscribed regular quindecagon. Ans. 12.475 feet.

61. If A_1, A_2 be the areas of two regular polygons, the former inscribed within, and the latter circumscribed about a circle; show that their difference is $A_1 \sin^2 \frac{\pi}{n}$.

62. In the last Problem, find the *least* value of n so that the difference of the areas of the two polygons may be less than the $\frac{1}{1000}$th part of the smaller. Ans. $n = 309$.

63. If a regular polygon, of an *even* number of sides, be inscribed in a circle, its area will be a mean proportional between

the areas of the inscribed and circumscribed polygons which have half the number of sides.

64. If a regular polygon of n sides be inscribed in a circle of radius R, and in the inscribed circle of this polygon a similar polygon be inscribed, and another similar polygon in the inscribed circle of this polygon, and so on, *ad inf.*; show that the sum of the areas of all the polygons will be $nR^2 \cot \frac{\pi}{n}$.

65. If the alternate sides of a regular polygon of n sides be produced to meet each other two and two, and if A_1, A_2 be the areas of the given polygon and of the new polygon; prove that

$$\frac{A_2}{A_1} = \frac{\cos^2 \frac{\pi}{n}}{\cos^2 \frac{2\pi}{n}}.$$

Show that n cannot be < 5.

66. In the last Problem, find the values of n

(1) when $A_2 = 3A_1$; (2) when $A_2 = \frac{\sqrt{5}}{4}(\sqrt{5} - 1) A_1$.

Ans. (1) $n = 6$; (2) $n = 10$.

67. If, in a regular polygon of n sides, diagonals be drawn from each point to every fourth point from it, taken in order, the polygon enclosed by these diagonals will be similar to the given polygon, and their areas will be in the ratio

$$\left(1 - 4 \sin^2 \frac{\pi}{n}\right)^2 : 4 \tan^2 \frac{\pi}{n}.$$

What is the least value which n can have? Ans. $n = 6$.

68. If all the shortest diagonals of a regular polygon of n sides be drawn, the polygon so formed will be similar to the given one, and their areas will be in the ratio of those in Problem 65.

69. Find the area of the whole surface of a triangular prism, each side of the base being 18 inches, and the length 10 feet. Ans. 46.949 sq. feet.

70. Find the area of the whole surface of a cube, each edge being 9 feet. Ans. 486 sq. feet.

71. The shortest diagonal of the base of a regular hexagonal prism is $2\sqrt{3}$ feet, and the altitude is 12 feet; find the area of the lateral surface. Ans. 144 sq. feet.

72. The area of the whole surface of a regular octagonal prism is 2070 square feet, and the area of the lateral surface is twice the area of the top; find the length of each side of the base. Ans. 10.35 feet.

73. The sides of a right prism, taken in order, are 3, 4, 5, 6 feet; the diagonal through the intersection of the first and last sides is 5 feet, and the altitude of the prism is 27 feet: find the whole area. Ans. 522 sq. feet.

74. The base of an oblique prism is a regular pentagon, each side being 3 feet; the perimeter of the transverse section is 12 feet, and the length of the prism 14 feet: find the area of the whole surface. Ans. 198.969 sq. feet.

75. An oblique prism, the transverse section of which is a regular hexagon of 3 feet in the side, is inclined to the horizon at an angle of 72°, and its vertical height is 10 feet; find the area of the inclined surface. Ans. 99.5 sq. feet.

76. The sides of the perimeter of the transverse section of a frustum of a prism are 3, 4, 5, 6 and 7 feet, and the lengths of the edges through the intersections of the first and second, second and third sides, &c. are 13, 17, 15, 14, and 12 feet respectively; find the area of the lateral surface.

Ans. 355.5 sq. feet.

77. A frustum of a regular octagonal prism is made by two planes inclined to each other at an angle of 30°; their line of intersection is parallel to the shortest face of the frustum, and one of these planes makes an angle of 45° with that face; the length of the shortest face is 7 feet, and the perimeter of the transverse section is 18.13 feet: find the area of the lateral surface. Ans. 163.17 sq. feet.

78. Find the area of the inclined surface of a square pyramid, each side of the base being 3 feet, and the slant height 15 feet.

Ans. 90 sq. feet.

79. Find the area of the whole surface of a triangular pyramid contained by four equal equilateral triangles, a side of each being 12 feet. Ans. 249.422 sq. feet.

80. A side of the base of a regular hexagonal pyramid is 5 feet, and the vertical height 15 feet; find the area of the inclined surface. Ans. 234.19 sq. feet.

81. Each side of the base of a regular octagonal pyramid is 4 feet, and the altitude is 20 feet; find the point in the axis through which a plane must be drawn parallel to the base to divide the lateral surface into two equal parts.

Ans. $10\sqrt{2}$ feet from the vertex.

82. The area of the surface of a frustum of a square pyramid is 100 sq. feet, the perimeter of the base is 13 ft. 4 inches, and the slant height 10 feet; find the area of the top.

Ans. 4 sq. feet 100 sq. inches.

83. The altitude of a frustum of an octagonal pyramid is 6 feet, and the perimeters of the base and top 40 and 32 feet respectively; find the area of the whole surface.

Ans. 589 sq. feet, very nearly.

Examples and Problems on Chapter II.

We have already given two examples of the application of the method of coordinates to determine the areas of plane rectilinear figures and the lengths of straight lines. We shall here show how any calculation may be verified by a simple change in either of the coordinate axes.

Suppose the axis of x to be moved, parallel to itself, through a unit of distance in the positive direction of the axis of y. Then the numerical values of all the ordinates parallel to the axis of y will be *diminished* by *one* when they are affected with a positive sign, and *increased* by *one* when they are affected with a negative sign.

1. Thus, in the Example in page 36, the new values of the ordinates will be $y_1 = 20.7$, $y_2 = 114$, $y_3 = 53$; and the calculation will stand thus :—

$$\begin{aligned} 2\text{ area} = {} & 83 \times 114 - 134 \times 20.7 \\ & + 134 \times 53 - (-61.5) \times 114 \\ & + (-61.5) \times 20.7 - 83 \times 53; \end{aligned}$$

$\therefore$ 2 area $= 23575 - 8445.85 = 15129.15$;

$\therefore$ area $= 7564.575$ sq. feet[*], as in the Example.

The student is recommended to verify the following Examples of areas in the manner we have explained.

2. Find the length of the straight line joining the points whose coordinates are

$$x_1 = 6.4,\ y_1 = 4.5, \text{ and } x_2 = -3.2,\ y_2 = -2.25;$$

and show that this line passes through the origin of coordinates and is trisected in that point.

3. The coordinates of the angular points of a triangle are $(0, 0)$, $(50.56, 25.28)$, $(50.56, -25.28)$; find the area of the triangle, and construct it. The triangle is isosceles, the vertex at the origin, and the base perpendicular to the axis of x.

4. The coordinates of the angular points of a triangle are $(10, 6)$, $(10, 6+2\sqrt{3})$, $(7, 6+\sqrt{3})$; construct the triangle, show that it is equilateral, and find its area.

5. The coordinates of the angular points of a quadrilateral field are $(10.25, 0)$, $(3.21, 7.35)$, $(-8.75, 0)$, $(2.37, -9.13)$ chains; find the area of the field in acres, and construct the figure. The axis of x is a diagonal.

6. The coordinates of the angular points of a field are $(0, 0)$, $(7.35, -3.5)$, $(10.15, 0)$, $(8.19, 7.21)$, $(-2.31, 5.17)$ chains; find its area and construct it. Also find the area of the same field from the following data:—the sides, taken in the same order as before, being 8.14, 4.48, 7.47, 10.32, 5.66; and the diagonals from the intersection of the first and last sides being 10.15, 10.91.

7. From the following entry in a note-book, as explained in

x		y	
1	11.21	1	3.47
2	24.30	2	4.32
3	31.61	3	10.00
4	28.12	4	19.59
5	16.07	5	22.33
6	7.31	6	19.61
7	4.42	7	8.23

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[*] There is a typographical error in page 36; the number 14939.15 should be 15129.15.

Art. 24, calculate the area of the figure and construct it; also find distance between the points (4) and (7). The figure is entirely clear of the axes.

8. The area of a plane rectilinear figure having been calculated from the coordinates of its angular points, and one of these coordinates having been lost; show how it may be recovered.

9. In the following form of a page of a note-book the suffices of x and y are entered only once, the corresponding values of the coordinates being entered on either side. Calculate the areas and construct the figures.

x		y
0	1	− 8.12
9.31	2	− 16.21
25.11	3	− 9.61
30	4	0
23	5	0
22.1	6	15.4
7.23	7	16.48
0	8	11

x		y
13.31	1	6.30
24.09	2	7.21
14.11	3	12.40
26	4	13.10
28.23	5	15
20	6	20.37
10	7	16.20
1.01	8	16
0	9	14.30

10. Calculate the area and construct the figure of the polygon, the coordinates of the angular points of which, in feet, are − (27, 0), (20, 3.75), (16.5, 4), (15, 7.5), (6, 6).

Examples and Problems on Chapter III.

1. The following equidistant ordinates of a plane curvilinear figure are measured from a straight line 215 feet in length; find the area, approximately, by each of the rules given in Chapter III:—

Ordinates—33, 42, 48, 50, 59, 66, 60, 57, 57, 58, 60, 51, 43, Here $n = 14$, $\therefore\ h = \frac{215}{14} = 15.36$, to two places of decimals; half-sum of extreme ordinates $= \frac{33+41}{2} = \frac{74}{2} = 37$; sum of intermediate ordinates $= 42 + 48 + \ldots + 43 + 40 = 691$.

Hence, by the First Rule,

$$\text{Area} = 15.36\ (37 + 691) = 11182.08 \text{ square feet.}$$

Again, for Simson's Rule, $2n = 14$, and $h = 15.36$, as before; sum of extreme ordinates $= 33 + 41 = 74$;

2 (sum of *odd* ordinates)

$$= 2\ (48 + 59 + 60 + 57 + 60 + 43) = 654;$$

4 (sum of *even* ordinates)

$$= 4\ (42 + 50 + 66 + 57 + 58 + 51 + 40) = 1456;$$

$$\therefore\ \text{Area} = \frac{15.36}{3}\ (74 + 654 + 1456) = 11182.08 \text{ sq. feet.}$$

It will be observed that the two results agree to the second place of decimals, or the areas, given by the two rules, differ by less than the hundredth part of a square foot, or by less than $1\frac{1}{2}$ square inches. It must not be concluded from this that the two rules are equally accurate. The reason of this close agreement in this particular example is easily explained by reference to the figure of Art. 25, the upper part of which, above the line *EF*, represents the figure we have been considering.

It will then be seen that, on account of the *undulating* nature of the curve *DC*, the straight lines, which complete the trapezoids, will fall, some within and some without the figure; and thus the *sum* of the trapezoids will agree, much more closely, with the area of the curvilinear figure than they would if the curve *DC* were *wholly concave* or *wholly convex* to the line *EF*. The figure on page 42 shows this compensation among the trapezoids very clearly, for it is obvious that the trapezoids, which are bounded on one side by *DL* and *LM*, are together very nearly equal to the whole curvilinear area in that figure.

We may conclude from these observations that the first rule is a very good one, when the curvilinear boundary is of an undulating nature.

2. The equidistant ordinates of a curvilinear area are 0, 40, 57, 60, 60, 56, 53, 45, 40, 32, 23, 15, 0; and they are measured from a base line 240 feet in length: find the area approximately.

Here the extreme ordinates are each zero; this shows that the base line is terminated both ways in the curvilinear boundary: this is very frequently the case in practice.

Here $n = 12$, $\therefore\ h = \frac{240}{12} = 20$ feet;

sum of extreme ordinates $= 0$;

sum of intermediate ordinates $= 481$;

$\therefore$, by First Rule, Area $= 20 \times 481 = 9620$ square feet.

Again, 2 (sum of *odd* ordinates) $= 466$;

4 (sum of *even* ordinates) $= 992$;

$\therefore$, by Simson's Rule, Area $= \frac{20}{3} \times 1458 = 9720$ square feet.

The two results differ by 100 square feet, about $\frac{1}{97}$th of the true area.

The curve, in this case, is wholly concave to the base line; and it will be obvious that the sum of the trapezoids will necessarily be less than the true area.

The results of the two Rules of Chapter III. may be tested in the manner already explained in the Examples of Chapter II. Thus, if in Ex. 1, all the ordinates be diminished by *one*, the whole area will be diminished by 215 sq. feet, whichever rule may be employed; and in Ex. 2, if the ordinates be increased by *one* (we could not diminish them in this case, without introducing negative signs, on account of the extreme ordinates being zero), the whole area will be increased by 240 sq. feet.

Thus, in Ex. 2, $\frac{1}{2}$ (sum of extreme ordinates) $= \frac{1+1}{2} = 1$;

sum of intermediate ordinates $= 492$;

$\therefore$ area $= 20 \times 493 = 9860$,

which differs from the former result by 240 square feet, as it ought.

3. From the following equidistant ordinates of a curvilinear figure, find its area, approximately, and construct the figure, the base line being 215 feet:—

Ordinates—30, 34, 39, 38, 33, 32, 37, 41, 42, 40, 38, 33, 30, 27, 29.

Area by First Rule = 7580.16; by Second Rule = 7562.26; and these differ by less than 18 square feet, or by less than $\frac{1}{420}$th part of the true area. The figure may be seen on page 40, being the portion of the figure, on that page, below the line *EF*.

Find, approximately, by both Rules, the areas of the curvilinear figures in the following examples:—

4. Ordinates—9, 11, 13, 12, 10, 13, 15, 17, 14, 12, 7: base = 73.

5. Ordinates—0, 9, 13, 17, 19, 22, 17, 13, 8, 4, 2: base = 61.

6. Ordinates—0, 3, 7, 11, 13, 18, 21, 20, 19, 17, 14, 9, 5, 2, 0: base = 147.

7. Ordinates—0, 9, 11, 10, 12, 15, 21, 24, 26, 22, 17, 15, 14, 12, 9, 6, 0: base = 166.

8. Find, approximately, by Simson's Rule, the area of a curvilinear figure from the following data:—

The base line measures 360 feet.

The positive ordinates are 0, 34, 56, 63, 71, 77, 80, 83, 90, 88, 81, 84, 70, 54, 0.

The negative ordinates are 0, 40, 56, 63, 74, 80, 77, 72, 69, 64, 62, 61, 55, 41, 0.

9. A meadow is enclosed on three sides by straight hedges, and on the remaining side by a stream: the straight line which joins the points where two of the hedges run down to the stream is taken for a base line, and measures 12.13 chains: fifteen equidistant ordinates measured from this line to the stream are as follows:—

0, 0.50, 1.51, 2.03, 1.91, 1.73, 1.73, 1.80, 2.0, 2.06, 1.84, 1.75, 1.30, 0.81, 0.

Also the coordinates of the two opposite corners are

2.16, −5.37; and 10.92, −4.13.

From these data find, approximately, the acreage of the meadow.

10. The figure on page 39 represents an irregular polygon obtained by setting up 10 poles in the boundary of an enclosure of about 50 acres. Seven equidistant ordinates are measured from each of the sides to the boundary, (those which are measured to the right hand of a line being considered positive, and those to the left hand negative, in going round the figure in the order 1, 2, 3, 4, 5, &c.), and their values are given in the following Table:—

Sides.	Values of equidistant ordinates.	Sign.
1—2	0, 21, 32, 24, 17, 15, 0,	−
2—3	0, 6, 17, 15, 13, 7, 0,	+
3—4	0, 3, 5, 6, 4, 3, 0,	+
4—5	0, 2, 3, 5, 4, 2, 0,	−
5—6	0, 17, 25, 24, 19, 11, 0,	+
6—7	0, 12, 14, 17, 15, 7, 0,	+
7—8	0, 3, 5, 7, 11, 9, 0,	+
8—9	0, 1, 2, 3, 2, 1, 0,	+
9—10	0, 10, 12, 15, 17, 9, 0,	+
10—1	0, 8, 11, 21, 23, 16, 0,	−

From these data, and the data of Art. 24, calculate, approximately, the area of the enclosure.

Note. The sides of the polygon must be calculated from the values of the coordinates given in Art. 24.

11. Required the area of a lake, and its outline, from the following data:—

Five poles, marked *A*, *B*, *C*, *D*, *E*, are set up to form a polygon which entirely encloses the lake. Each side of the polygon is divided into 10 equal parts, and 9 equidistant ordinates are measured from the points of division to the margin of the lake, no ordinates being measured from the poles themselves. The sides and angles of the polygon, and the values of the equidistant ordinates are given in the following Table, the sides in chains, and the ordinates in links.

Side.		Angle at point.		Equidistant ordinates.								
AB	8·33	*B*	120^0	61,	33,	10,	18,	57,	62,	44,	30,	22.
BC	7·79	*C*	112^0	56,	64,	79,	95,	82,	70,	53,	27,	99.
CD	7·34	*D*	100^0	76,	60,	84,	126,	93,	55,	61,	82,	106.
DE	11·71	*E*	109^0	99,	72,	56,	131,	123,	104,	90,	47,	99.
EA	8·60	*A*	190^0	91,	42,	13,	77,	121,	98,	86,	42,	75.

The small polygons, at the corners of the great polygon, must be taken into account: they may be divided into two, or three small triangles, at the most, with sufficient accuracy for the approximation.

12. Apply Simson's Rule to determine, approximately, the area of a quadrant of a circle, the radius being a unit of length; employing 11 equidistant ordinates, the first being the radius of the circle, and the last zero. The ordinates may be calculated from the formula

$$y = \sqrt{1 - x^2},$$

y being the ordinate, and x its distance from the centre.

Ans. 0.78175.

The result, true to 6 decimals, is 0.785398.

Examples and Problems on Chapter IV.

1. The coordinates of a parapet, taken as in Art. 29, are

$$x_1 = 6'.\ 4'',\ x_2 = 16'.\ 4'',\ x_3 = 17'.\ 5'',\ x_4 = 20'.\ 5'',\ x_5 = 27'.\ 7'';$$

$$y_1 = 6'.\ 4'',\ y_2 = 8'.\ 0'',\ y_3 = 3'.\ 9'',\ y_4 = 3'.\ 7'',\ y_5 = 0'.\ 0'';$$

$'$ denoting *linear* feet, and $''$ *linear* inches; find the area of the section.

Taking the areas of the triangles separately, we find, by duodecimals,

2 area of $BAC = 6'.4'' \times 8'.0'' - 16'.4'' \times 6'.4'' = 52$ sq.ft. $9'.\ 4''$;
2 area of $CAD = 16'.4'' \times 3'.9'' - 17'.5'' \times 8'.0'' = 78$ sq.ft. $1'.\ 0''$;
2 area of $DAE = 17'.5'' \times 3'.7'' - 20'.5'' \times 3'.9'' = 14$ sq.ft. $1'.10''$;
2 area of $EAF = 20'.5'' \times 0'.0'' - 27'.7'' \times 3'.7'' = 98$ sq.ft. $10'.\ 1''$;

sq. ft., $'$, $''$, in the last three columns, denoting respectively, square feet, primes (or twelfths of square feet), and square inches (or twelfths of primes);

∴ 2 area of section = 243 sq. ft., 10 primes, 3 sq. inches;

∴ area of section = 121 sq. ft., 11′. $1\frac{1}{2}''$ = 122 square feet, very nearly.

The same result may be obtained by dividing the section into three trapezoids and two triangles, as in the figure of Art. 29. Also, by computing the values of GB', HE', and KD', and thence finding KG, as explained in Art. 30.

Thus we find $KG = 30'.\ 5\frac{3}{4}''$, very nearly, and multiplying this by 8′, we find

2 area = 243 sq. feet, 10 primes.

2. Required the depth of a ditch, the transverse section of which is a trapezoid, the breadth of the top 20 feet, the slopes of the sides 2 in 1, and 3 in 1, and the area of the section 146.25 square feet.

By Art. 31, we have

$$A = xd - \frac{m+n}{2mn} d^2.$$

Here $A = 146.25$, $x = 20$, $m = 2$, $n = 3$;

$$\therefore\ 146.25 = 20d - \frac{5}{12} d^2,$$

$$\therefore\ d^2 - 48d + 351 = 0;$$

$$\therefore\ d^2 - 48d + (24)^2 = 576 - 351 = 225,$$

$$\therefore\ d - 24 = \pm 15,$$

$$\therefore\ d = 24 \pm 15 = 39 \text{ or } 9.$$

Hence the required depth is 9 feet.

The answer 39 may be explained as follows:—

The sides AC, BD (see fig. Art. 31) would necessarily meet at a point 24 feet below AB; and the answer 24 ± 15 shows that the line CD is to be drawn either 15 feet *above* or 15 feet *below* this point: if we consider the area *below* this point to be *negative*, the area of the section, in each case, will be 146.25 sq. feet.

3. Example on Art. 32.

Given $A_2 = 122$ square feet, $a = 31$ feet, $h = 8$ feet, $d = 9$ feet,
$m = 4$, $n = 3$, $p = 1$;

slope of $CD = 1$ in 6,—and ∴ $SC' = a + b = 48$ feet,—required x, the breadth of the ditch at top.

The general equation being

$$xd - \frac{m+n}{2mn} d^2 = A_2 + \frac{h}{2} \frac{(a-x)^2}{a+b+\frac{h}{p}},$$

we have, in this case,

$$9x - \frac{7}{24} 81 = 122 + 4 \frac{(31-x)^2}{56},$$

or

$$9x - \frac{7 \times 27}{8} = 122 + \frac{(31-x)^2}{14};$$

$$\therefore 404x - 1323 = 6832 + 4(961 - 62x + x^2);$$

$$\therefore 4x^2 - 652x = -11999;$$

$$\therefore x^2 - 163x = -2999.75;$$

$$\therefore x^2 - 163x + (81.5)^2 = 6642.25 - 2999.75$$
$$= 3642.5;$$

$$\therefore x - 81.5 = \pm\sqrt{3642.5} = \pm 60.25;$$

$$\therefore x = 81.5 \pm 60.25 = 21.25,$$

the other value of x being inadmissible.

Hence the breadth to be set out for the top of the ditch is 21 feet 3 inches.

4. The coordinates of the transverse section of a Ravelin being

0, 7; 6, 13; 24, 16; 25.5, 11.5; 30, 11; 36, 8.5; 63, 8:

construct the section and find its area;—the axis of x being the intersection of the plane of the transverse section with the plane of site.

5. Find the area of the section of a parapet from the following values of its coordinates, the axes being taken as in Ex. 4,

0, 0; 17, 17; 35, 20; 36.5, 15.5; 41, 15; 48, 12; 75, 11; 80, 8.5; 90, 8; 96, 0.

The section in this case belongs to the Prussian system.

6. Find the area of the section of a ditch, and the breadth at top; the depth being 10 feet, the breadth at bottom 11.5, the slopes of the sides 2 in 1, and 3 in 1, and the section a trapezoid.

7. Find the breadth of a ditch at top; the area of the section being 160 sq. feet, the depth 9 feet, the slopes 3 in 1, and 4 in 1, and the section a trapezoid.

8. The slopes of the sides of a ditch being 5 in 2, and 7 in 2, and the breadth at top 22 feet; how deep must the ditch be that the sides may meet at the bottom?

9. Find the top breadth of a ditch 10 feet deep, slopes of sides 1 in 1; so that the earth excavated from it may form a parapet of the following dimensions (see fig. Art. 29):

$$x_1=6,\ x_2=16,\ x_3=17.5,\ x_4=20.5,\ x_5=27;$$

$$y_1=6,\ y_2=7.5,\ y_3=3.5,\ y_4=3.5,\ y_5=0.$$

10. In the last problem, find the breadth, allowing one-tenth for excess of earth.

11. The slopes of a ditch in front of a field-work are 1 in 1, and they meet at the bottom: behind the work is a trench 9 feet wide at the top, 3 feet wide at the bottom, and 3 feet deep. The coordinates of the parapet are 0, 0; 4, 4; 10, 5; 11.5, 1; 15, 1; 17, 0; measured as in Art. 29. Find the breadth and depth of the ditch so that the earth taken out of the ditch and trench may be just sufficient to form the parapet.

12. Taking the data of Example 3, find what depth must be given to the ditch if the breadth of the top be 19 feet 6 inches.

13. If one-twelfth be allowed for excess of earth, find the value of the breadth in Example 3, and of the depth in Example 12.

14. A field-work is constructed on a slope of 1 in 10, the direction of the parapet being at right angles to the line of steepest descent: $A_2=135$ square feet, $b=19$ feet, the slope of CD is 1 in 5, the remaining dimensions (excepting a) being as in Example 3; find the breadth required for the top of the ditch.

Examples and Problems on Chapter V.

1. A RECTANGULAR block of marble is 4 ft. 6 in. long, 3 ft. 2 in. wide, and 2 ft. 6 in. deep; its specific gravity (or the ratio of its weight to the weight of an equal volume of distilled water) is 2.638; and the weight of a cubic foot of distilled water is 62.5 lbs. very nearly: find the weight of the block.

Here volume $= 54 \times 38 \times 30$ cubic inches,

$$= \frac{54 \times 38 \times 30}{12^3} \text{ cubic feet};$$

and the weight of a cubic foot of the marble is

$$2.638 \times 62.5, \text{ or } 164.88 \text{ lbs.};$$

$$\therefore \text{ weight of block} = \frac{54 \times 38 \times 30 \times 164.88}{12^3} \text{lbs.}$$

$$= \frac{2 \times 6^3 \times 2 \times 19 \times 3 \times 10 \times 2 \times 10.305}{3^3 \times 2^6} \text{ lbs.}$$

$$= 19 \times 3 \times 103.05 \text{ lbs.} = 5873.85 \text{ lbs.}$$

The same by Duodecimals:

$$\begin{array}{l}
4 \text{ ft. } 6 \text{ in.} \\
\ \ 3 \text{ ft. } 2 \text{ in.} \\
\hline
13 \text{ sq. ft. } 6' \\
\ \ 0 \text{ sq. ft. } 9' \\
\hline
14 \text{ sq. ft. } 3', \text{ or } 14 \text{ sq. ft. } 3 \text{ primes,} \\
\ \ 2 \text{ ft. } 6 \text{ in.} \\
\hline
28 \text{ cu. ft. } 6' \\
\ \ 7 \text{ cu. ft. } 1'.\ 6'' \\
\hline
35 \text{ cu. ft. } 7'.\ 6'',
\end{array}$$

or 35 cubic feet, 7 primes, 6 seconds; $'$ denoting primes (or twelfths of cubic feet), and $''$ seconds (or twelfths of primes); and, similarly, $'''$ would denote thirds (or twelfths of seconds), and therefore cubic inches*.

---

* Compare with Example 1, p. 223.

Now the weight of one cubic foot is, as before, 164.88 lbs.;

$$\therefore \text{ weight of } 35 \text{ cu. ft.} = 164.88 \times 35 = 5870.80 \text{ lbs.};$$

$$\text{weight of } 7' = 164.88 \times \frac{7}{12} = 96.18 \text{ lbs.};$$

$$\text{and weight of } 6'' = 164.88 \times \frac{6}{12^2} = 6.87 \text{ lbs.};$$

$$\therefore \text{ weight of block} = 5873.85 \text{ lbs., as before.}$$

Hence the block weighs 2.62225 tons, or rather more than $2\frac{1}{2}$ tons.

The student who is acquainted with the subject of "Notation," in Algebra, will prefer the following method of performing the multiplications:—

4:6 representing "4 duodecimal 6," the process will stand thus:—

$$\begin{array}{rrrrr}
 & & & 4: & 6 \\
 & & & 3: & 2 \\
\cline{4-5}
 & & & 9 & 0 \\
 & 1 & 1 & 6 & \\
\cline{2-5}
 & 1 & 2: & 3 & 0 \\
 & & 2: & 6 & \\
\cline{2-5}
 & 7 & 1 & 6 & 0 \\
2 & 4 & 6 & 0 & \\
\cline{1-5}
2 & e: & 7 & 6 & 0,
\end{array}$$

e representing *eleven* in the scale of which the radix is *twelve;*

$$\therefore \text{ volume} = 2 \times 12 + 11 + \frac{7}{12} + \frac{6}{12^2} = 35 \text{ cu. ft. } 7 \text{ primes } 6 \text{ seconds.}$$

2. A frustum of a square pyramid is 10 feet in vertical height, and the sides of the base and top are 3 ft. 6 in., and 2 ft. 3 in. respectively; find the volume.

By Art. 40,

$$V = \frac{h}{3}\left(A_1 + \sqrt{A_1 A_2} + A_2\right).$$

Here $h=10$, $A_1=(3.5)^2=12.25$, $A_2=(2.25)^2=5.0625$;

$$\therefore \sqrt{A_1 A_2}=\sqrt{(3.5)^2\times(2.25)^2}=3.5\times 2.25=7.875;$$

$$\therefore V=\frac{10}{3}(12.25+7.875+5.0625)$$

$$=\frac{10}{3}(25.1875)=83.9583 \text{ cu. ft.}$$

Hence the volume $=84$ cubic feet, very nearly.

3. The transverse section of a wedge is an isosceles triangle, the sides of which are double of the base, and the perimeter is 25 inches; also, the three parallel edges of the wedge are 15, 17, and 22 inches: find its volume.

By the formula of Art. 41,

$$V=A\,\frac{h_1+h_2+h_3}{3},$$

Here $A=\sqrt{12.5\times 2.5\times 2.5\times 7.5}$ Art. 6.

$$=2.5\sqrt{93.75}=2.5\times 9.68=24.2;$$

and $\frac{h_1+h_2+h_3}{3}=\frac{15+17+22}{3}=\frac{54}{3}=18;$

$$\therefore V=24.2\times 18=435.6 \text{ cubic inches.}$$

4. An embankment is made upon a slope of 1 in 10; the top of the embankment is horizontal and its section everywhere a trapezoid; its greatest height above the slope is 57 feet, the breadth of the top 26 feet, and the slopes of the sides 1 in 1: find the number of cubic yards in a length of 160 yards of the embankment.

The solid in question is evidently a prismoid; and, employing the notation of Art. 46, and the fig. 2, we have

$$AD=a=26,\ AY=p_2=57,\ \therefore\ FY=57;$$

$$\therefore FG=c=2\times 57+26=140.$$

Also $AX = p_1 = 57 - XY = 57 - \frac{l}{10} = 57 - \frac{480}{10} = 9$;

$$\therefore EX = 9, \text{ and } EH = b = 2 \times 9 + 26 = 44.$$

Hence we know a, b, c, p_1, p_2;

$$\therefore A_1 = \frac{1}{2}(a+b)\,p_1 = \frac{1}{2}(26+44)\,9 = 315 \text{ sq. ft.};$$

$$A_2 = \frac{1}{2}(a+c)\,p_2 = \frac{1}{2}(26+140)\,57 = 4731 \text{ sq. ft.};$$

$$4A = \frac{1}{2}(2a+b+c)(p_1+p_2)$$

$$= \frac{1}{2}(52+44+140)(9+57) = 118 \times 66 = 7788 \text{ sq. ft.}$$

Hence we obtain

$$6V = 480\,(315 + 4731 + 7788) \text{ cubic feet,}$$

$$\therefore V = 80 \times 12834 = 1026720 \text{ cubic feet};$$

$$\text{or } V = 370266 \text{ cu. yds. } 18 \text{ cu. ft.}$$

5. The base of a prismoidal solid is a square, and the top a regular octagon, four alternate sides of which are parallel to the sides of the base. The altitude of the solid is 6 feet, the sides of the base 3.5 feet, and those of the top 1 foot; find its volume.

By the Prismoidal Formula

$$6V = l\,(A_1 + A_2 + 4A).$$

Here $l = 6$; $A_1 = (3.5)^2 = 12.25$; $A_2 = 4.828427$. Ex. 4, p. 205.

Also the mean section will be an eight-sided figure, the alternate sides of which will be equal to each other; and if x represent one of the four greater sides, y one of the four remaining sides,

$$x = \frac{1 + 3.5}{2} = \frac{4.5}{2} = 2.25;\quad y = \frac{1}{2} = .5.$$

(This will easily be seen if a *plan* of the solid be drawn.)

Hence the mean section is the difference of two squares, the side of one of which is $(2.25 + .5\sqrt{2})$ feet, and of the other $.5$ feet;

$$\therefore A=\left(\frac{4.5+\sqrt{2}}{2}\right)^2-\left(\frac{1}{2}\right)^2=\frac{(5.5+\sqrt{2})\,(3.5+\sqrt{2})}{4};$$

$$\therefore 4A=19.25+9\sqrt{2}+2=21.25+12.728=33.978.$$

Hence $$6V=6\,(12.25+4.828+33.978);$$

$$\therefore V=51.056 \text{ cubic feet.}$$

6. A room is 40 feet long, 22 feet wide, and 15 feet high; how many cubic yards does it contain?

7. A rectangular cistern is 12 feet long, 9 feet wide, and 4 feet deep; how many gallons will it contain, if 277.274 cubic inches are equivalent to one gallon?

8. How much deeper must the cistern be made if it be required to hold 125 gallons more?

9. If a be the length of each edge of a cube, shew that the diagonal of each face is $a\sqrt{2}$, and the diagonal of the solid $a\sqrt{3}$.

10. A box, in the form of a rectangular parallelepiped, with a lid, has a uniform thickness of three-fourths of an inch; and the outside measurements are 23, 15, and 17.5 inches: how many cubes, each half an inch in the edge, will it contain?

11. If 277.274 cubic inches are equivalent to one imperial gallon, find the length of the edge of a cube which will contain exactly 10 gallons.

12. If one brick occupy a space 9 inches by $4\frac{1}{2}$ by 3, how many would be required for a wall 100 feet in length, 10 feet in height, and a brick and a half in thickness?

13. Each edge of a cube is diminished by one-tenth of its length; by how much is the volume diminished?

14. The height of a solid six-inch cube is diminished by pressure to $5\frac{5}{8}$; supposing the lateral expansion uniform throughout the mass, what will be the dimensions of the new base of the solid?

15. A horse-trough is 15 feet long, 2 feet wide at the top, 1 foot wide at the bottom, and 1 foot deep; how many gallons will it contain when full?

If 100 gallons be withdrawn, how many inches will the water sink?

16. In the last problem, if 10 gallons flow into the trough in 3 minutes, and 9 gallons flow out in 4 minutes; how long will it take to fill the trough; and how much water will have passed through it before it is filled?

17. Water, 1 foot deep, flows through a rectangular trough 4 feet wide, at the rate of 5 miles per hour; how many cubic feet of water will pass a given point in one minute?

18. Three cubes of metal, whose edges are 3, 4, and 5 inches respectively, are melted and formed into a single cube; if there be no waste in the process, show that the edge of the new cube will be 6 inches.

19. In the last problem, if one-tenth of the metal be wasted in the process, find the length of the edge of the new cube.

20. A log of English oak is 15 feet long, 18 inches bread, and 12 inches thick; at what distance from one end must the log be cut, that the smaller portion may weigh 5 cwt., supposing one ton of English oak to contain 36.205 cubic feet?

21. The base of a parallelepiped is a rectangle, the sides of which are 7 and 11 inches; and the altitude of the solid is 8 inches: find the volume.

22. The base of a parallelepiped is a parallelogram two sides of which are 13 and 17 inches, and the angle between them 60^0: find the volume, if the altitude of the solid be 9 inches.

23. In the last problem, if the angle between the sides of the base be reduced to 30^0, what must the altitude be that the volume may remain the same?

24. If the altitude and the sides of the base of a parallelepiped, the volume of which is V, be all increased in the same proportion $\left(\frac{1}{p} \text{ suppose}\right)$; show that the increase in the volume will be expressed by

$$V\left\{\left(1+\frac{1}{p}\right)^3-1\right\}.$$

Ex. If the linear increase be one-tenth, the volume will be increased by one-third, very nearly.

25. The section of a canal is 32 feet wide at the top, 14 feet wide at the bottom, and 8 feet deep; how many cubic yards were excavated in a mile of the canal? Also, if the surface of the water be 26 feet wide, what is its depth?

26. The sides of a triangular prism are 3, 4, and 5 inches, and the length 15 inches: find the number of cubic inches in the volume.

27. The transverse section of a prism is an isosceles triangle, the angles at the base being double of the third angle; the perimeter of the section is 12 inches, and the length of the prism 22 inches: find the volume.

28. The back of a wedge is 15 inches by 9; and the height 12 inches: find the volume.

29. A roof has a rise of 1 in $2\frac{1}{2}$; the span is 30 feet and the length 72: find the cubical content.

30. The base of a prism is a parallelogram whose sides are 7 and 9 inches, and the angle between them 60°; and the altitude of the prism is equal to the sum of the diagonals of the base: find the volume.

31. The transverse section of a prism is a pentagon, the sides of which, taken in order, are 10, 13, 19, 13, and 9 inches, and the two diagonals measured from the intersection of the first and last sides are 20 and 19 inches; the length of the prism is 5 feet: find its volume in cubic feet and inches. Also, if the altitude of the prism be 4 ft. 3 inches, find the area of the base.

32. Find the volume of a regular hexagonal prism, the central diagonal of the base of which is 18 inches, and the length 15 feet.

33. The transverse section of an oblique prism is a regular octagon, the perimeter being 10 feet; and the area of the base is 8.7114 square feet: show that the axis of the prism is inclined to its base at an angle of 60°.

34. A square pyramid is 10 feet high, and the sides of the base are each 4 feet; find its volume. At what distance from the vertex would a horizontal plane bisect the solid?

35. The slanting edges of a triangular pyramid are each 12 feet, and each side of the base is 6 feet; find the volume.

36. A regular hexagonal pyramid has the perimeter of its base 15 feet, and its altitude 15 feet; find its volume.

37. The walls of an octagonal room are each 15 feet wide, and 15 high; the roof is an octagonal pyramid having a rise in the centre of 10 feet: find the cubical contents of the room.

38. The representative gold pyramid in the International Exhibition of 1862 was 10 ft. square at the base, and 44 ft. $9\frac{1}{4}$ inches in height; find the volume in cubic feet: also the weight, if one cubic inch of gold weigh 10.14502 oz. troy; and the value, at 80s. per oz.

Ans. Volume $= 1492.36\dot{1}$ cu. ft.; weight $= 26,162,432$ ounces Troy; or 800 tons 17 cwt. 3 qrs. 7 lbs. 3.12 oz. Avoirdupois.

Value $= £104,649,728$.

39. It was stated on the pyramid, considered in the last problem, that the gold-fields of Victoria had yielded an additional quantity, since the construction of the pyramid, to the value of £4,642,328: how much higher would the pyramid require to be made to include this quantity?

Ans. 2 feet, *very nearly.*

40. The base of the Great Pyramid of Egypt is a square of 764 feet in the side, and the altitude 477.6 feet; find the volume in cubic yards, neglecting the inequalities in the surface.

41. A frustum of a square pyramid of marble is 5 feet high, 3 feet in the side of the base, and 2 feet in the side of the top; find its weight if one cubic foot of the marble weigh 164.88 lbs.

42. A triangular pyramid is contained by four equal equilateral triangles; required to cut off one-third of it by a plane parallel to the base.

43. A frustum of a pyramid has the area of its base four times that of the top: show that its volume is seven-twelfths of that of a prism which has an equal base and altitude.

44. The altitude of a frustum of an octagonal pyramid is 20 feet; each side of the base is 6 feet, and each side of the top 4 feet: and the frustum is surmounted by an octagonal pyramid the altitude of which is 5 feet: find the volume of the whole solid.

45. The areas of the base and top of a frustum of a pyramid being given, and also the altitude, it is required to bisect the frustum by a plane parallel to the base.

46. Taking the data of the last problem, determine the position of a plane, parallel to the base, which shall divide the solid into two parts having to each other the ratio $m:n$; and verify the result by applying it to the preceding problem.

47. Find the volume of a frustum of a triangular prism the transverse section of which is an equilateral triangle having a perimeter of 15 inches; the sum of the three parallel edges being 45 inches.

48. A ditch is to be made 100 yards long, 10 feet wide at the top, and 5 feet deep; and the slopes of the sides and ends everywhere 1 in 1: find the number of cubic yards to be excavated.

49. The roof of a house has a rise of 2 in 5; and the eaves measure 70 feet in length by 45 feet in width: find the number of square yards of slating, and the content of the roof in cubic yards.

[From the formula for the volume of a frustum of a triangular prism we may easily deduce a general formula for determining the number of shot in a pile, whether the pile be rectangular, triangular, or square.

For this purpose, suppose the transverse section of the solid to be an equilateral triangle, the base a rectangle, and the two triangular faces to be inclined to the base at an angle of 60°: also that the numbers of units of length in each of the three parallel edges, the number of units of area in the transverse section, and the number of units of volume in the solid, are all of them *whole* numbers.

Let N be the number of units of volume; and conceive the solid to be divided into N equal cubes, and each of these cubes to become a sphere. These spheres will manifestly form a rectangular pile having the same *relative* proportions as the solid from which they were formed.

Hence, if in the formula $V = A\,\dfrac{h_1+h_2+h_3}{3}$, we replace V by N, A by F (F being the number of shot in one of the triangular faces), and $h_1+h_2+h_3$ by E, the number of shot in the three parallel edges; the resulting formula,

$$N=F\frac{E}{3},$$

will determine the number of shot in the pile. We shall deduce the usual formulæ from this.

Let n be the number of courses; then, for the rectangular pile,

$$E=p+2(p+n-1)=3p+2n-2,$$

p being the number of balls in the ridge.

For the square pile, $E=n+n+1$, the ridge consisting of one ball.

For the triangular pile, $E=n+1+1$, the ridge and one of the edges at the base consisting of one ball each.

Also, in all the piles, $F=1+2+3+\ldots+n=n\dfrac{n+1}{2}$.

Hence the three resulting formulæ will be as follow:—

Rectangular pile, $N=\frac{1}{6}n(n+1)(3p+2n-2)$;

Square pile, $N=\frac{1}{6}n(n+1)(2n+1)$;

Triangular pile, $N=\frac{1}{6}n(n+1)(n+2)$.

These formulæ are well known: they are the particular cases of the general and much more simple formula

$$N=F\frac{E}{3}.$$

Ex. A complete rectangular pile consists of 15 courses, and there are 20 balls in the ridge: find the number in the pile.

Here $F=1+2+3+\ldots+15=\dfrac{15\times16}{2}=120,$

$E=20+2(20+15-1)=20+68=88;$

$\therefore N=\dfrac{120\times88}{3}=40\times88=3520.]$

50. A rectangular parallelepiped is cut by two planes which do not meet the ends: if h_1, h_2, h_3, h_4 be the lengths of the four parallel edges, A the area of the transverse section, and V the volume of the frustum; show that

$$V = \frac{A}{4}(h_1 + h_2 + h_3 + h_4).$$

51. The base of a prism is a regular pentagon of one foot in the side; and a frustum is made by a plane passing through a point in the axis of the prism 12 feet above the base: find the volume. Will the volume be altered by turning the plane about the given point in any manner?

52. A regular hexagonal prism is cut by a plane inclined to the base at an angle of 45°, and the distance of the line of intersection of the cutting plane with the plane of the base from the centre of the base is three times its perimeter: find the volume of the frustum, a side of the base being 2 feet.

53. The perimeter and altitude of a regular prism being given, determine the position of a plane which shall pass through one of the sides of the top, and cut off one-third of the volume.

If θ be the inclination of the cutting plane to that of the top, $\tan\theta = \frac{2n}{3p} h \tan\frac{\pi}{n}$, h being the altitude, p the perimeter, and n the numbers of sides of the prism.

54. In the last problem, if the cutting plane pass through one of the angular points of the top, and be parallel to the shortest diagonal which subtends the angle at that point; determine the inclination in that case.

55. An octagonal stone prism stands at the foot of a sloping bank of grass which is inclined to the horizon at an angle of 45°; the line of intersection of the slope with the ground is parallel to one face of the prism, and one foot in advance of it. If the prism be 8 feet high, and 12 feet in perimeter, what proportion of its volume is above the bank?

56. Find the volume of a frustum of a prism from the following data:—The transverse section is a quadrilateral whose sides, taken in order, are 7, 5, 4, 8 inches, and the diagonal through the intersection of the first and last is 7.5 inches: also

the lengths of the edges, taken in the same order, from the same angular point, are 22, 25, 23, and 21 inches.

57. A frustum of a prism has its transverse section an irregular hexagon, the sides of which, taken in order, are 14.3, 11, 7.6, 12, 10.7, 11.8 inches; and the diagonals through the intersection of the first and last are 18.7, 21.8, 21.5 inches: also the lengths of the edges, taken in the same order, from the same angular points, are 65, 72, 77, 79, 81, 70 inches: find the volume in cubic feet.

Note. The figure, page 75, represents the transverse section.

58. Taking the data of each of the last two problems, and supposing one end of the frustum to be perpendicular to the edges, determine the position of a parallel plane which shall bisect the solid.

59. Find the volume of a portion of a parapet from the following data: the dimensions of the transverse section are given in Ex. 1. page 223; and the "lines on the plan,"—x_1, x_2, &c., of Art. 45,—are 102′, 84′, 79′.6″, 82′.6″, 81.6″, and 92′.3″.

60. The principal traverse in the covered way of the Modern system of fortification has one face vertical, and the opposite face inclined to the horizon. The dimensions of the vertical face, taken as in Art. 29, are 6′, 24′, 25′.6″, 30′, 37′.6″; and 5′, 8′, 4′.9″, 4′.6″, 0′. The length of the traverse at the base is 36′, and the length of the crest line 33′. Compute the lengths of the other "lines on the plan," and find the volume of the traverse.

61. A line of parapet, having the same transverse section as the traverse in the last problem, is contained between two vertical planes inclined to the crest line at angles of 30° and 70°; and the length of the crest line is 75 yards. Find the volume of the parapet.

62. A prismoid is 20 feet in length, and the breadth of the base is 3 feet. The breadth of the top at one end is 5 feet, and at the other end 7.5 feet; the depth at the smaller end 6 feet, and at the opposite end 13.5 feet. Find the volume of the prismoid.

63. A prismoidal solid is cut from a prism, the transverse section of which is an equilateral triangle, so that the smaller

end shall be a regular hexagon having its perimeter two-thirds of that of the prism. Find the volume of the solid in terms of the perimeter and length of the prism.

64. The base of a regular prismoidal solid is a dodecagon, and the top a hexagon, and the solid is surmounted by a pyramid 4 feet in height. The perimeter of the base is 15 feet, that of the top 9 feet, and the altitude of the whole 25 feet. Find the volume.

65. An irregular prismoidal solid has its base a square, and its top a rectangle having half the area of the base. The diagonal of the top is equal to a side of the base, and the altitude of the solid is three times the diagonal of the base. Find the volume in terms of a side of the base.

66. Find the number of solid feet of timber in a tree the ends of which are rectangles, the sides of the greater being 15 and 12 inches, and the corresponding sides of the other 7 and 5 inches; the length of the tree being 33 feet 6 inches.

67. Find the capacity of a coal-waggon the top of which measures 6 feet 9 inches in length, by 4 feet 6 inches in breadth, the bottom 3 feet 6 inches by 2 feet 6 inches, and the depth 4 feet.

68. A watercourse, having a fall of 1 foot in 400 yards, is to be cut in a straight line on level ground. The sides are to slope 1 in 1, the bottom to be 6 feet wide, and the depth at the upper end 4 feet. Find the number of cubic yards in the first mile of the trench; and the expense of cutting it if the cost be 3*d.* per cubic yard for the first foot of depth, 4*d.* for the second foot, and so on; 1*d.* per cubic foot being added for each foot of depth.

69. Find the number of cubic yards of earth excavated from a railway-cutting made through ground the original surface of which was an inclined plane running in the same direction as the rails; the length of the cutting being 4 chains 15 links, the breadth at bottom 30 feet, the breadth at top at one end 75 feet, and at the other 135, and the depths of these ends 20 feet and 46 feet respectively.

70. A railway-embankment is to be made on ground which falls 20 feet in a mile in the direction of the rails; and the rails

themselves fall 1 in 700. The embankment is straight for $2\frac{1}{4}$ miles; the breadth at the top 29 feet, the slopes of the sides 1 in 1, and the height at the upper end 3 feet. Find the number of cubic yards of earth required for the embankment.

71. Find the number of cubic yards of earth in a portion of a railway-cutting 16 chains in length; the following numbers representing the areas, in square yards, of a series of transverse sections taken at intervals of 1 chain:—300, 291, 284, 276, 290, 273, 261, 250, 237, 221, 204, 187, 171, 196, 204, 215, 220.

72. A railway-cutting is to be made 30 feet wide at the bottom, and the slopes of the sides $1\frac{1}{2}$ in 1. The depths in feet on one side of the cutting, taken at intervals of 1 chain, are 12, 11, 10, 9, 10, 12, 14, 15, 17, 20, 21, 23, 25, 27, 30, 33, 37, 41, 45, 50, 53; and the corresponding depths on the opposite side 15, 14, 13, 12, 14, 15, 17, 19, 22, 26, 28, 30, 33, 36, 39, 41, 44, 51, 57, 60, 58: find the number of cubic yards of earth to be excavated.

The areas of the transverse sections may be computed from the formula $A = \frac{h}{2}(p_1 + p_2) + \frac{2}{3} p_1 p_2$, h being the breadth of the cutting at bottom, and p_1, p_2 corresponding opposite depths.

73. An embankment is made upon a curve of 40 chains radius, the length measured along the centre line of the roadway being 12 chains. The areas of twenty-five equidistant transverse sections, computed from the dimensions of the embankment, taken at intervals of half a chain, are as follows: find the volume in cubic yards.

Areas in square feet,—175, 193, 206, 221, 254, 276, 301, 331, 335, 346, 357, 381, 396, 404, 421, 447, 423, 415, 401, 387, 356, 330, 315, 291, 256.

74. An embankment runs in a straight line, for half a mile, across a slope which has an inclination of 20^0 to the horizon; and the direction of the embankment makes a horizontal angle of 70^0 with the steepest line of descent. The greatest height of the embankment above the slope, on the centre line of the roadway, is 54 feet, its breadth at top 30 feet, and the slopes of its sides 1 in 1: find the number of cubic yards of earth in it.

75. Two curved portions of a railway embankment, each having a radius of 60 chains, and made on level ground, meet in a point (a cusp) in the line joining their centres of curvature. Supposing the top of the embankment to be 33 feet above the ground, the breadth of the top 30 feet, and the slopes of the sides $1\frac{1}{2}$ in 1; find the volume contained between a vertical plane passing through the centres of curvature and two other planes passing through these points and inclined to the former at angles of 45^0.

76. In the last problem, if the ground fall 1 in 20 in the direction of a line perpendicular to the line joining the centres of curvature; find the volume, the other data remaining as before.

Examples and Problems on Chapter VII.

1. A gravel-walk, of uniform width, is to be made round a circular basin of large dimensions, filled with water; find the area of the walk from dimensions taken with a yard measure; and the diameter of its outer circumference.

Since a yard measure only is to be used, we cannot measure the diameter of the basin directly. We must therefore measure the breadth of the walk and *some other line* on the ground. The most convenient line will be that which is at once a chord of the outer circumference of the walk, and a tangent to the inner circumference.

Let $2a$ feet be the length of this line, and h the breadth of the walk; and suppose R and r the radii of its outer and inner circumferences;

$$\therefore\ a^2 = R^2 - r^2.$$

Also $\quad$ area of walk $= \pi R^2 - \pi r^2 = \pi\,(R^2 - r^2) = \pi a^2.$

Hence the area of the walk is known when we know the length of the measured chord $2a$.

To find the diameter of the outer circumference of the walk, we have

$$a^2 = R^2 - r^2 = (R + r)\ (R - r) = (2R - h)\ h,$$

since $R - r = h$;

$$\therefore\ R = \frac{a^2 + h^2}{2h};$$

$$\therefore \text{ diameter} = 2R = \frac{a^2 + h^2}{h}.$$

2. The chord of a segment of a circle measures 120 feet, the versed sine 12 feet, and the arc 156 feet; find the area of the segment.

By the last formula of Art. 71, we have

$$\text{area} = \frac{a^2 (c - a) + h^2 (c + a)}{2h}.$$

Here $a = 60$, $c = 78$, $h = 12$;

$$\therefore \text{ area} = \frac{3600 \times 18 + 144 \times 138}{24}$$

$$= 300 \times 9 + 138 \times 6$$

$$= 2700 + 828;$$

$$\therefore \text{ area} = 3528 \text{ square feet.}$$

The simplicity of the calculation in this case shows the advantage of *measuring* the arc whenever practicable: the length of the arc can, of course, be calculated from the chord and versed sine; but this calculation is always tedious and troublesome.

3. If $2a$ be the clear span, h the rise, h' the thickness of a circular arch, and A the area of its face; show that

$$A = \frac{h'}{h} (a^2 + h^2 + hh') \sin^{-1} \frac{2ah}{a^2 + h^2};$$

and find the numerical value of A when $2a = 100$, $h = 30$, and $h' = 4.5$ feet.

The face of the arch will evidently be a sector of a circular annulus: hence, by the formula of Art. 72, if $2C$ and $2c$ represent the lengths of the upper and under surfaces of the arch,

$$A = (2C + 2c) \frac{h'}{2} = (C + c) h'.$$

Now let 2θ be the angle subtended at the centre by the arch, and r the radius of its under surface;

$$\therefore C = (r + h') \theta, \text{ and } c = r\theta;$$

$$\therefore A = h' (2r + h') \theta.$$

Now $$a^2 = h (2r - h) = 2rh - h^2;$$

$$\therefore\ r=\frac{a^2+h^2}{2h}.$$

Also $$\sin\theta=\frac{a}{r}=\frac{2ah}{a^2+h^2};$$

$$\therefore\ \theta=\sin^{-1}\frac{2ah}{a^2+h^2}.$$

Hence $$A=h'\left(\frac{a^2+h^2}{h}+h'\right)\sin^{-1}\frac{2ah}{a^2+h^2},$$

or $$A=\frac{h'}{h}(a^2+h^2+hh')\sin^{-1}\frac{2ah}{a^2+h^2}.$$

To find the numerical value of A when $2a=100$, $h=30$, and $h'=4.5$ feet.

By the formula just proved, we have

$$A=\frac{4.5}{30}(2500+900+30\times 4.5)\sin^{-1}\frac{3000}{2500+900}.$$

$$=\frac{3}{20}(2500+900+135)\sin^{-1}\frac{30}{34}$$

$$=530.25\sin^{-1}\frac{15}{17}\text{ square feet.}$$

Now the symbol $\sin^{-1}\frac{15}{17}$ represents the circular measure of the angle whose sine is $\frac{15}{17}$; let x be the number of degrees, minutes and seconds in this angle, a its circular measure;

$$\therefore\ \frac{a}{\frac{\pi}{2}}=\frac{x}{90},\ \frac{\pi}{2}\text{ being the C. M. of a right angle.}$$

Now $\sin x=\frac{15}{17}$; $\therefore$ Log $\sin x=10+\log 15-\log 17$

$=9.9456424$

$=$ Log $\sin 61^\circ\,55'\,39''$;

$$\therefore\ x=61^\circ\,55'\,39'',\text{ or }61^\circ.9275.$$

$$\therefore\ a = \frac{\pi}{2}\frac{x}{90} = \frac{\pi}{180}\,61.9275$$

$$= \frac{3.1416 \times 61.9275}{180} = 1.08,$$

to two places of decimals.

Hence $A = 530.25 \times 1.08 = 572.67$ sq. feet.

4. A lune is formed by the intersection of two equal circles, the circumference of each passing through the centre of the other: find the area of the lune and the length of the common chord.

Let r be the radius of the circles, $2c$ the common chord, and 2θ the angle which it subtends at the centre of each circle.

Then, Art. 73, area of lune $= r^2(\pi - 2\theta + \sin 2\theta)$.

Now by the geometry of the figure, we have

$$\cos\theta = \frac{\frac{r}{2}}{r} = \frac{1}{2};\quad \therefore\ \theta = \frac{\pi}{3};\ \text{and } \sin\theta = \frac{\sqrt{3}}{2}.$$

$$\therefore\ \sin 2\theta = 2\sin\theta\cos\theta = \frac{\sqrt{3}}{2}.$$

$$\text{Hence area of lune} = r^2\left(\pi - \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) = r^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right).$$

To find the length of the chord, we have

$$c = 2r\sin\theta = 2r\,\frac{\sqrt{3}}{2} = r\sqrt{3}.$$

5. The minor axis of an ellipse is taken for the latus-rectum of a parabola, and one of the foci for its vertex: compare the areas of the two portions into which the ellipse is divided by the parabola; and find the excentricity of the ellipse that the problem may be possible.

Let $2a$, $2b$ be the major and minor axes of the ellipse, $2p$ the distance between the foci, and A_1, A_2 the areas of the two portions of the ellipse, A_1 being the greater.

Then
$$A_1 = \frac{\pi ab}{2} + \frac{2}{3}(2b)p = \frac{\pi ab}{2} + \frac{4}{3}pb;$$

$$A_2 = \frac{\pi ab}{2} - \frac{2}{3}(2b)p = \frac{\pi ab}{2} - \frac{4}{3}pb.$$

Now by a property of the ellipse

$$p^2 = a^2 - b^2;$$

and by a property of the parabola

$$p = \frac{b}{2};$$

$$\therefore\ p^2 = a^2 - 4p^2;\ \therefore\ 5p^2 = a^2, \text{ and } p = \frac{a}{\sqrt{5}}.$$

Hence,
$$\frac{A_1}{A_2} = \frac{\dfrac{\pi ab}{2} + \dfrac{4ab}{3\sqrt{5}}}{\dfrac{\pi ab}{2} - \dfrac{4ab}{3\sqrt{5}}} = \frac{3\pi\sqrt{5}+8}{3\pi\sqrt{5}-8}.$$

Again, let e be the excentricity of the ellipse,

$$\therefore\ e^2 = 1 - \frac{b^2}{a^2} = \frac{a^2 - b^2}{a^2} = \frac{p^2}{a^2} = \frac{1}{5}.$$

Hence, the problem will be possible (that is, a parabola may be described having one focus of the ellipse for its vertex, and the minor axis for its latus-rectum) when the excentricity of the ellipse is $\frac{1}{\sqrt{5}}$; and the ratio of the areas A_1, A_2 will be as above.

6. A perfectly flexible rope, of uniform thickness, is coiled closely upon a level floor. If the diameter of the rope be $2a$, show that the length of the first n complete coils will be

$$\pi an(2n+1).$$

In order that the coil may be accurately formed, we shall suppose the extremity of the rope to be a hemisphere.

It will then be easily seen by a figure that the first complete coil will consist of two semi-circular portions whose radii are a and $2a$; the second, of two portions whose radii are $3a$ and $4a$; and so on.

$$\begin{aligned}
\text{Hence the length of the 1st coil} &= \pi a + 2\pi a,\\
\text{.............. 2nd ...} &= 3\pi a + 4\pi a,\\
\text{.............. 3rd ...} &= 5\pi a + 6\pi a,\\
\text{........................} &= \text{............}\\
\text{.............. } n^{\text{th}} \text{ ...} &= (2n-1)\pi a + 2n\pi a;
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ length of first } n \text{ coils} &= \pi a\,(1+2+3+\ldots+2n)\\
&= \pi a\,(2n+1)\,\frac{2n}{2}\\
&= \pi an\,(2n+1).
\end{aligned}$$

We have given this example at length because the curve in question is not obviously a case of the involute of a regular polygon, since no polygon can have less than three sides. If, however, we suppose a straight line to have two sides, and consider it as a *regular polygon of two sides*, the general formula which we have obtained in Art. 77 (1), will be applicable to the present example.

By that formula, we have

$$\text{length of first } r \text{ coils} = \pi ar\,(rn+1);$$

and here we have to replace r (the number of coils) by n, and n (the number of sides of the polygon) by 2, when we obtain

$$\text{length of first } n \text{ coils} = \pi an\,(2n+1),$$

the same result as before.

If the coil commence at some distance from the centre, its length may be found by taking the difference of two coils, each commencing at the centre. And if the number of coils be not a whole number, the fraction of a coil above a whole number of coils must be taken into account separately.

There are many applications of the example we have considered; such as a watch-spring, closed; a roll of paper; an electro-magnetic coil of wire, where the coil is all in one plane. The case of a coil of wire upon a cylinder is a particular case of a spiral spring, the general case of which has been investigated in Art. 99.

7. Show that the area of the $(r+1)^{\text{th}}$ convolution of the involute of a regular hexagon is double of the area of the $(2r+1)^{\text{th}}$ convolution of the involute of an equilateral triangle, if the perimeter of the hexagon be double of that of the triangle.

Since the perimeters are as 2 to 1, the sides of the triangle and hexagon are equal.

Now, by Art. 77, the area of the $(r+1)^{\text{th}}$ convolution of the involute of a regular polygon of n sides is

$$\pi a^2 n\,(2rn+1),$$

a being the length of a side.

Hence, for the hexagon,

$$\text{area of } (r+1)^{\text{th}} \text{ convolution} = \pi a^2 6\,(2r6+1)$$
$$= \pi a^2\,(72r+6);$$

and for the equilateral triangle,

$$\text{area of } (2r+1)^{\text{th}} \text{ convolution} = \pi a^2 3\,\{2\,(2r)\,3+1\}$$
$$= \pi a^2\,(36r+3).$$

Now since $$72r+6=2(36r+3),$$

whatever r may be, the area in the one case is double of that in the other.

Thus, the area of the 3rd convolution of the involute of the hexagon is twice the area of the 5th convolution of the involute of the triangle; that of the 4th twice that of the 7th, and so on.

8. Determine the angle through which the string, which describes the involute of a circle, must revolve, so that the area of the involute may be twenty times that of the circle.

For this problem we shall employ the formula of Art. 79, viz.—

$$\text{Area} = \frac{1}{6a}\,(R\phi_1{}^3 - R\phi^3),$$

where a is the radius of the circle, ϕ_1 the whole angle described by the string, $R\phi_1$ the length of the string unwound, and $\phi=\phi_1-2\pi$.

Now $$R\phi = a\phi,\ R\phi_1 = a\phi_1 = a\,(\phi+2\pi)$$

$$\therefore\ \text{Area} = \frac{a^3}{6a}\,\{(\phi+2\pi)^3-\phi^3\}.$$

Hence, by the conditions of the problem,

$$\frac{a^2}{6}\,\{(\phi+2\pi)^3-\phi^3\} = 20\pi a^2;$$

$$\therefore\ 3\phi^2 2\pi + 3\phi\, 4\pi^2 + 8\pi^3 = 120\pi.$$

$$\therefore\ \phi^2 + 2\pi\phi = 20 - \frac{4}{3}\pi^2;$$

$$\therefore\ \phi^2 + 2\pi\phi + \pi^2 = 20 - \frac{\pi^2}{3}$$

$$= 20 - 3.2898681$$

$$= 16.7101319;$$

$$\therefore\ \phi + \pi = \pm 4.0878;$$

$$\therefore\ \phi_1 = \phi + 2\pi = \pi \pm 4.0878;$$

$$\therefore\ \phi_1 = 7.22939.$$

This last number represents the circular measure of the angle: the number of degrees will therefore be between 413° and 414°. Hence the string must describe one complete revolution, and, in addition, an angle greater than 53° and less than 54°.

9. The diameter of a circle is 10 feet 3 inches; find the area true to a square inch; and the circumference true to a tenth of an inch.

10. The circumference of a circle being 10 chains; find the diameter of another circle which has half the area of the former.

11. If a circle contain one acre, show that its diameter will be 1 chain 56.8 links.

12. Find the cost of turfing a circular grass-plot of 50 feet in diameter, at *6d.* per square yard.

13. The circumference of the circular basin of a fountain measures 171 feet on the outside of the masonry, and the thickness of the masonry is 30 inches: find the area of the surface of the water.

14. An obstacle stands over the centre of a circle so that a diameter cannot be measured; find the area of the circle from three chords which form a triangle.

15. Compare the area of an equilateral triangle with that of a circle which circumscribes it.

16. Compare the area of a square with that of a circle which circumscribes it.

17. Compare the areas of the circles inscribed within and circumscribed about an equilateral triangle.

18. If a circle be inscribed in an equilateral triangle, an equilateral triangle in this circle, another circle in this triangle, and so on continually; find the limit of the sum of the areas of all the circles.

19. Find the area of a sector of a circle, the radius being 3 feet, and the angle 120°.

20. The arc of a sector is 33 feet, and the radius 30 feet; find the area.

21. Find the angle of a sector the radius of which is 60 feet, and the area 125 square feet.

22. Two sectors have equal areas, and their radii are as 2 to 1; show that their angles are as 4 to 1.

23. If the area of a sector be 99 square feet, and the length of the arc 9 feet; find the radius, and the angle at the centre.

24. The chord and versed sine of a sector being 27 and 9 inches respectively; find its area.

Find the area of a segment of a circle—

25. When the radius is 3 feet, and the angle at the centre 60°.

26. When the radius is 30 inches, and the angle at the centre 36°.

27. When the radius is 10 feet, and the circular measure of the angle 2.4.

28. When the chord is 20 feet, and the versed sine 10 feet.

29. When the chord is 125 feet, and the angle at the centre 112°. 30′.

30. When the chord is 100, and the radius 80 feet.

31. When the arc measures 125 feet, and subtends an angle of 120°.

32. Show that the chord of a quadrant divides a circle into two parts, the areas of which are in the ratio of 10 to 1, very nearly.

33. Three equal circles intersect each other, so that the circumference of each passes through the centres of the other two: find the area of the figure common to the three circles.

34. Three equal circles touch each other; show that the

area of the space between them is $r^2\left(\sqrt{3}-\frac{\pi}{4}\right)$, r being the radius of the circles.

35. Three circles, all unequal, touch each other externally: find the area of the space between them.

36. If A be the area of a triangle, and r, r_1, r_2, r_3 the radii of the inscribed and three escribed circles; show that

$$A^2 = r\, r_1 r_2 r_3.$$

37. The chord of a segment of a circle measures 141.42, the versed sine 29.29, and the arc 157.08 feet: find the area of the segment.

38. A circular arch of 100 feet span rises 13.4 feet, and the length of the arc is 104.72 feet: find the area of the headway above the chord line.

39. A circular arch has a clear span of 140 feet, and a clear height of 29 feet: the thickness at the crown is 5 feet, and at the abutments 6 feet 3 inches. The back of the arch is also an arc of a circle, and measures 164 feet 6 inches: find the area of the face.

40. A pointed arch is formed by two equal circular segments, the radius of each being equal to the span; find the area of the headway.

41. Two equal segments of a circle form a pointed arch whose clear span is 50 feet, rise 30 feet, and length 85 feet; find the area of the headway.

42. A band passes round two wheels, and crosses between them, and is kept stretched: if the distance between the centres of the wheels be a, the sum of their diameters $2c$; show that the length of the band will be

$$2\left\{\sqrt{a^2-c^2}+c\left(\pi-\cos^{-1}\frac{c}{a}\right)\right\}.$$

Also find the whole area enclosed by the band.

43. Find the length of a band which just passes round three wheels, of unequal radii, whose centres are at the angular points of an equilateral triangle; and find the whole area enclosed by the band.

44. Find the area of the zone of a circle, contained between

two parallel chords whose lengths are 96 and 60 inches, and their distance apart 26 inches.

Area = 2136.753 sq. inches, to three places of decimals.

45. Find the area of a zone of a circle whose parallel sides are 20 and 15 inches, and their distance 17.5 inches.

Area = 395.437 sq. feet, to three places of decimals.

46. If the diameters of two concentric circles which form an annulus be 20 and 12 inches, its area will be 201 square inches, very nearly.

47. The outer diameter of an annulus is 9 feet, and the breadth 9 inches: find the area.

48. If c be the length of a chord of an annulus, which also touches the inner boundary; show that the area $= \pi \dfrac{c^2}{4}$.

49. If ρ be the mean radius of an annulus, and b its breadth; show that the area $= 2\pi\rho b$.

50. A circular hole is to be cut in a circular plate, so that the weight may be reduced one-third: find the diameter of the hole.

51. Three men have equal shares in a grindstone 3 feet in diameter, and they agree not to grind down more than 9 inches in all: what part of the diameter may each man grind down?

52. A sector of an annulus is 7 inches broad, and its bounding arcs are 25 and 21 inches: find its area.

53. If a circular arch, composed of 5 rings of brickwork, have a clear span of 30 feet, and measure 37 feet on the back; find its rise, and the area of its face, supposing the depth of each brick to be 4½ inches.

54. Two circles whose radii are 12 and 9 inches intersect, and their common chord is 12 inches; find the areas of the two lunes formed by them.

55. The common chord of two circles which intersect subtends an angle of 60° at the centre of one circle, and an angle of 120° at the centre of the other; and the length of the chord is 15 inches: find the radii of the circles and the areas of the lunes.

56. One circle passes through the extremities of a diameter of another, and bisects a radius at right angles; find the areas of the lunes formed by the circles.

57. Two equal circles, which have a radius of 12 inches, have a common chord of 9 inches: find the area of either lune.

58. If the area of the lune formed by the intersection of two equal circles be half that of either circle; and if $2c$ be the length of the common chord, and r the radius; prove that

$$\sin^{-1}\frac{c}{r}-\frac{c}{r}\sqrt{1-\frac{c^2}{r^2}}=\frac{\pi}{4}.$$

59. If semicircles be described upon the sides of a right-angled triangle, the sum of the areas of the two lunes thus formed will be equal to the area of the triangle.

60. Find the area of the space which is common to four equal circles which intersect each other, their centres being at the angular points of a square and their radii equal to a side of the square.

61. Find the circumference and area of an oval, constructed as in Art. 74, the length of AB being 22 feet.

62. Construct an oval from the following data, and find its circumference and area:—

Take a horizontal line $AB=30$ feet; bisect it in E, and through E draw an indefinite vertical line. Take EG (upwards) $=7$, $GH=4$; and EF (downwards) $=8$ feet. Join AG, BG, AF, BF, and produce them indefinitely beyond the points G and F. In GA, GB take GK, GK' each equal to 5 feet. Join KH, $K'H$, and produce them indefinitely beyond H: and in AB take AL equal to 25 feet. With centre A and radius AL describe a circular arc terminated by AF and AG produced, and with centres F, B, K', H, and K describe a series of circular arcs so as to form with the first a continuous curve.

63. From the following data a section of the Kilsby Tunnel on the London and North Western Railway may be constructed and its area found:—

Take AB a horizontal line $18'$ in length; bisect it in O, and through O draw an indefinite vertical line. Take OC (upwards) $=3'.6''$, $CF=3'.6''$, and $FG=9'$; and OH (downwards) $=11'.4''$. Join AC and in it take $CK=CF$, and produce AC and KF indefinitely. With centre F and radius FG describe a circular arc, and with centres K and A describe other arcs forming with the first a continuous curve: with center G and distance GH describe an arc forming the base of the tunnel, and intersecting the former curve in a finite angle. The ground line is $4'$ above the point H.

64. Find the area of the clear headway of a semi-oval arch, constructed as follows, the span being 120 feet:—

Take AB horizontal and equal to 120 feet; bisect it in E, and draw EG vertically downwards, and equal to EA. Bisect EG and EA in F and H; join FH, and in it take $HK=HE$, and join GK. With centres H, K, G, and two other centres symmetrical with K and H about the line EF, construct the semi-oval terminated at A and B.

65. Find the area of an ellipse whose axes are 30 and 20 inches.

66. The major axis of an ellipse is 37 inches, and the distance between its foci 19 inches: find the area.

67. An ellipse has its minor axis 15 feet in length, and its eccentricity $\frac{1}{3}$; find its area.

68. Find the area of an elliptic annulus formed by two similar concentric ellipses, whose axes are given.

69. Find the area of an elliptic lune formed by two ellipses which have a common major axis but different minor axes.

70. Find the excentricity and area of an ellipse which contains two-thirds of the area of the circle on its major axis.

71. Show that the area of any ellipse is a geometric mean between the areas of the circles on its major and minor axes.

72. If the area of the double lune contained between an ellipse and the circle on its major axis be twice the area of the double lune contained between the same ellipse and the circle on its minor axis; show that the ratio of the axes will be 2 : 1.

73. The clear span of a semi-elliptic arch is 96 feet, and the clear height 32 feet. The thickness of the arch at the crown is 4 feet, and at the springing 5 feet 6 inches: find the area of the face.

74. The axes of an ellipse being 45 and 57 inches; find the area of a segment cut off by a line parallel to the minor axis, the height of the segment being 10 inches; and compare the area with that of a segment, of the same altitude, cut off by a line parallel to the major axis.

75. An elliptic segment is cut off by a line parallel to the minor axis: the height of the segment is 5 inches, its base 10 inches, and the major axis of the ellipse 20 inches; find the area of the segment.

76. If the minor axis of an ellipse be the major axis of a similar ellipse, and the minor axis of this ellipse the major axis of another similar ellipse, and so on continually; show that the sum of the areas of all these ellipses *ad. inf.* will be

$$\frac{\pi a^3 b}{a^2 - b^2},$$

a and b being the semi-axes of the given ellipse.

77. A segment of a parabola is cut off by a straight line, perpendicular to the axis, 15 inches in length, and the height of the segment is 4.5 inches: find its area.

78. Find the area of the segment of a parabola cut off by its latus-rectum, the length of which is 34 inches.

79. The base and altitude of a segment of a parabola, cut off by a chord perpendicular to the axis, being given; it is required to bisect it by a parallel chord.

80. If a circle touch a parabola at the extremities of its latus-rectum, find the area of the parabolic lune contained between the circle and the parabola.

81. If a circle be described on the latus-rectum of a parabola as diameter, find the area of the parabolic lune contained between the circle and the parabola; the latus-rectum being given.

82. A parabolic arch has a clear span of 30 feet, and a clear height of 45 feet, and its latus-rectum is 20 feet; if the thickness of the arch at the crown be 3 feet, find the area of its face.

83. If p_1, p_2 be two chords of a parabola, parallel to each other and perpendicular to the axis, and if z be their distance; show that the area of the parabolic zone between them will be

$$\frac{2}{3} z \frac{p_1^2 + p_1 p_2 + p_2^2}{p_1 + p_2}.$$

84. Find the area of a segment of a parabola, cut off by a focal chord inclined to the axis at an angle of 60°, the latus-rectum of the parabola being 12 inches.

85. If a be the latus-rectum, and θ the inclination of the chord to the axis; find a general expression for the area of segment, and apply it to the preceding Example.

86. A focal chord of a parabola is trisected in the focus; find the area of the segment cut off by it, the latus-rectum being given.

87. An ellipse and a parabola have one focus common, and the other focus of the ellipse is in the vertex of the parabola; also, the latus-rectum of the parabola is equal to the major axis of the ellipse: find the areas of the two portions of the ellipse, one on either side of the parabola.

88. Find the area of the first convolution of the involute of an equilateral triangle whose perimeter is 15 inches.

89. Find the length and area of the third convolution of the involute of a square whose perimeter is 20 inches.

90. The area of the seventh convolution of the involute of a regular pentagon is 25 square feet; find the length of a side of the pentagon.

91. Find the length and area of the involute of a regular pentagon, after the string has made 10 revolutions, the perimeter of the pentagon being 11 feet.

92. Find the length and area of the involute of a regular decagon, after the string has been unwrapped from 31 consecutive sides of the polygon, the perimeter of the polygon being 15 inches.

93. The sides of a regular polygon are 2 inches in length; and the area of the involute after the string has made 11 revolutions is 236.7 square feet: find the number of sides of the polygon.

94. In the involute of a regular polygon, if A_{r+1} denote the area swept over by the string during its $(r+1)^{\text{th}}$ revolution; show that

$$A_{r+1} - 2A_r + A_{r-1}$$

is constant for the same polygon.

95. Deduce the general expressions for the length and area of the involute of a circle, in Art. 79, from the data of Art. 78.

96. If the radius of a circle be 6 inches, and the length of the string unwound 36 inches; find the area of the involute described. Will the string have made more or less than one revolution?

97. Find the area and length of the involute of a circle whose diameter is 10 inches, after the string has made 5 complete revolutions: also find the length and area of the fifth convolution of the same involute.

98. The string which describes the involute of a circle makes 3.2 revolutions, and the length of the string then unwound is 84 inches. If the angle described by the string be now increased by 300^0, find the area of the portion of the involute bounded by the first and last positions of the string.

99. Find the length of the involute of a circle, after the string has described 6 revolutions and an angle of 36^0, the length of the string unwound being 94.75 inches.

100. If C_{r+1} denote the length of the $(r+1)^{\text{th}}$ convolution of the involute of a circle, and c the circumference of the circle; show that

$$C_{r+1} - C_r = 2\pi c.$$

101. Find the length and area of a cycloid, the diameter of the generating circle being 4 feet 2 inches.

102. A straight line is drawn through the centre of the generating circle of a cycloid, parallel to its base: find the length of the arc of the cycloid above this line.

103. Find the area contained between a cycloid and its evolute, the diameter of the generating circle being given.

104. In the figure, page 128, if AB, AC be taken for the semi-axes major and minor of an ellipse; compare the area of the ellipse with that of the cycloid.

105. If cycloids be described on the sides of a right-angled triangle as bases; shew that the area of the greatest will be equal to the sum of the areas of the other two.

106. A cycloidal arch is contained between two cycloids which have their bases and axes coincident; the clear span of the arch is 74 feet, and the thickness at the crown 33 inches: find the area of the face of the arch.

107. Find the area of a segment of a cycloid, cut off by a normal at any point in the curve.

108. A straight line is drawn through a given point in the axis of a cycloid, parallel to its base: find the area of the seg-

ment cut off by it. If the point be at a distance $\frac{3}{4}AC$ above A, fig. of Art. 80, show that the area of the segment $=\frac{3}{16}AC^2\sqrt{3}$.

109. If the normal which bisects a semicycloid be inclined to the axis of the cycloid at an angle a; show that

$$2a+\sin 2a=\frac{\pi}{2}.$$

110. Find the area of a segment of a cycloid, cut off by a straight line, parallel to the axis, passing through a given point in the base.

Examples and Problems on Chapter VIII.

1. If a spherical shell, when formed into a solid sphere, be equal in volume to its own cavity; what must be the thickness of the shell?

Let V be the volume of the shell, R its radius, and r the radius of the cavity.

Then $$V=\frac{4}{3}\pi(R^3-r^3).\quad \text{Art. 94.}$$

Also V is the volume of the cavity,

$$\therefore\ V=\frac{4}{3}\pi r^3.$$

Hence $R^3-r^3=r^3$; $\therefore\ R^3=2r^3$, and $r=\dfrac{R}{\sqrt[3]{2}}$.

Now the thickness of the shell is $R-r$;

$$\therefore\ \text{thickness of shell}=R-\frac{R}{\sqrt[3]{2}}$$

$$=R\left(1-\frac{1}{\sqrt[3]{2}}\right)$$

$$=\frac{R}{2}\sqrt[3]{4}\,(\sqrt[3]{2}-1).$$

Substituting the values of $\sqrt[3]{4}$ and $\sqrt[3]{2}$ we find

$$\text{thickness of shell}=R\times 0\cdot 20629.$$

2. A hollow cylinder of cast iron, 20 feet in length, and 6 feet in diameter outside, is placed on end and loaded uniformly on the top with a weight of 300 tons; determine the thickness of the metal so that the pressure on the base may be one ton per square inch; the weight of a cubic foot of the iron being 441 lbs.

Let t be the thickness of the cylinder, expressed in feet, w its weight in tons.

Therefore $w + 300$ is the number of tons pressure on the base, and equal to the number of square inches in the area of the base.

$$\text{Now area of base} = \pi\{3^2 - (3-t)^2\} \text{ square feet}$$

$$= 144\pi(6t - t^2) \text{ square inches;}$$

$$\therefore \text{ volume of cylinder} = 20\pi\,(6t - t^2) \text{ cubic feet;}$$

$$\therefore \text{ weight of cylinder} = 20 \times 441\pi\,(6t - t^2) \text{ lbs.}$$

$$= \frac{20 \times 441}{2240}\pi\,(6t - t^2) \text{ tons.}$$

Hence we have

$$144\,\pi\,(6t - t^2) = 300 + \frac{20 \times 441}{2240}\pi\,(6t - t^2);$$

$$\therefore \left(144 - \frac{441}{112}\right)\pi\,(6t - t^2) = 300.$$

Taking $\pi = \frac{22}{7}$, this equation becomes, after reduction,

$$t^2 - 6t = -\frac{2100}{3168};$$

$$\therefore t = 3 \pm 2.8874.$$

The positive sign is obviously inadmissible; therefore the value of t is 0.1126 feet, $= 1.5312$ inches.

Hence the thickness is rather more than $1\frac{1}{2}$ inches.

We may now find the weight of the cylinder by substituting the value of t (in feet) in the equation

$$w + 300 = 144\pi\,(6t - t^2);$$

for which we obtain $\qquad w = 18.225$ tons.

3. A heavy sphere is placed in a hollow circular cone whose axis is vertical and vertex downwards. The intermediate space is filled with water, which just covers the sphere as it begins to flow over. If the sphere be withdrawn slowly, find the depth through which the water will sink.

Let 2α be the vertical angle of the cone, and h its altitude; r the radius of the sphere, and x the depth through which the water sinks when the sphere is withdrawn.

We shall have

$$r = (h-r)\sin\alpha\,;\ \therefore\ r = \frac{h\sin\alpha}{1+\sin\alpha}.$$

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\frac{h^3\sin^3\alpha}{(1+\sin\alpha)^3}.$$

Now the space left empty when the sphere is withdrawn is a frustum of a cone, the radii of its ends being $h\tan\alpha$, and $(h-x)\tan\alpha$, and its altitude x.

Therefore volume of this space

$$= \pi\frac{x}{3}\{h^2\tan^2\alpha + (h-x)^2\tan^2\alpha + h(h-x)\tan^2\alpha\}$$

$$= \frac{\pi}{3}x\tan^2\alpha\,\{h^2 + h(h-x) + (h-x)^2\}\,;$$

and the volume of this space is equal to the volume of the sphere;

$$\therefore\ x\tan^2\alpha\,(3h^2 - 3hx + x^2) = \frac{4h^3\sin^3\alpha}{(1+\sin\alpha)^3}.$$

This is a cubic equation which will determine x when h and α are known.

Ex. Let $h = 10$ inches, $\alpha = 30^0$;

$$\therefore\ \sin\alpha = \frac{1}{2},\ 1+\sin\alpha = \frac{3}{2},\ \tan^2\alpha = \frac{1}{3};$$

$$\therefore\ \frac{x}{3}(3\times 10^2 - 3\times 10x + x^2) = \frac{4\times 10^3\times\frac{1}{8}}{\left(\frac{3}{2}\right)^3};$$

$$\therefore \frac{x^3}{3} - 10x^2 + 100x = \frac{10^3}{2} \times \left(\frac{2}{3}\right)^3 = \frac{4000}{27};$$

$$\therefore x^3 - 30x^2 + 300x - \frac{4000}{9} = 0.$$

This equation must have at least one positive root; and, from the conditions of the problem, we see that x must lie between 0 and r, that is, between 0 and $\frac{10}{3}$; $x=0$ makes the left-hand member negative, and $x=2$ makes it positive; therefore the root in question lies between 0 and 2. It may be found by approximation to be 1.77936.

Hence the water will sink through 1.8 inches, very nearly.

4. A spherical shell of given material floats in water: if the radius of the shell be given, and the depth of the segment immersed; show how to find the thickness of the shell.

Ex. A cast-iron shell, 12 inches in diameter, and 10 inches immersed, the specific gravity of the iron being 7.05.

Let R be the radius of the shell, x its thickness, and h the depth of the segment immersed.

S the specific gravity of the material, S' that of the water.

Then $$\text{weight of shell} = \frac{4}{3}\pi S\{R^3 - (R-x)^3\}.$$

Again, the water which the shell displaces is a segment of a sphere, the radius of which is R, and the height h;

$$\therefore \text{weight of water displaced} = \pi S' h^2 \left\{R - \frac{h}{3}\right\}.$$

Now when a body floats, at rest, in a fluid, the weight of the fluid which the body displaces is equal to the weight of the whole body*: hence we have

$$\frac{4}{3} S\{R^3 - (R-x)^3\} = S'h^2 \left\{R - \frac{h}{3}\right\}.$$

This is a cubic equation for determining x when R, h, S and S' are given.

In the Example, $R=6$, $h=10$, $S=7.05$; and, if the water be distilled or rain water, $S'=1$.

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* This principle must be assumed in all problems of this nature.

$$\therefore \frac{4}{3} \times 7.05\{216-(6-x)^3\}=100\left(6-\frac{10}{3}\right);$$

$$\therefore 1522.8-7.05(6-x)^3=200;$$

$$\therefore (6-x)^3=\frac{1522.8-200}{7.05}=\frac{1322.8}{7.05};$$

$$\therefore 3 \log(6-x)=\log 1322.8-\log 7.05=2.2733051;$$

$$\therefore \log(6-x)=0.7577684=\log 5.7249;$$

$$\therefore 6-x=5.7249;$$

$$\text{or } x=0.2751 \text{ inches.}$$

5. A cylindrical pontoon with hemispherical ends is constructed of sheet iron $\frac{1''}{8}$ in thickness; the extreme length of the pontoon is 20′ 8″, and its outside diameter 2′ 8″. Find the weight which the pontoon will support when half immersed, and the greatest load which it can bear; assuming the specific gravity of the iron to be 7.7, and the weight of a cubic foot of water to be 63 lbs.

Let W be the weight which the pontoon supports when half immersed, W_1 the weight of the pontoon, and W_2 the weight of the water which it displaces;

then $$W_2=W_1+W \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

And if W' be the greatest weight which the pontoon can support,

$$2W_2=W_1+W' \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now area of section of pontoon

$$=\pi\left(16+16-\frac{1}{8}\right)'' \times \frac{1''}{8}=\pi\frac{255}{64} \text{ sq. inches};$$

$$\therefore \text{volume of cylindrical portion}=\pi\frac{255\times 18\times 12}{64\times 12^3} \text{ cubic feet};$$

and by Art. 94, (1),

$$\text{volume of hemispherical ends}=4\pi 16^2\times\frac{1}{8} \text{ cubic inches,}$$

$$=\pi\frac{16^2}{2\times 12^3} \text{ cubic feet.}$$

∴ volume of iron in the pontoon

$$= \frac{\pi}{12^3}\left\{\frac{255 \times 18 \times 12}{64} + \frac{16^2}{2}\right\} \text{ cubic feet};$$

$$\therefore\ W_1 = \frac{63\pi}{12^3} \times 7.7\left\{\frac{255 \times 27}{8} + 128\right\} \text{ lbs.}$$

$$= \frac{63\pi}{12^3} \times 7612.4125 \text{ lbs.} = 871.91 \text{ lbs.}$$

Again, volume of water displaced $= \frac{1}{2}$ volume of pontoon.

$$\therefore\ W_2 = \frac{63\pi}{12^3}\left\{9 \times 16^2 \times 12 + \frac{2 \times 16^3}{3}\right\} \text{ lbs.}$$

$$= \frac{63\pi}{12^3} \times 30379.\dot{6} \text{ lbs.} = 3479.58 \text{ lbs.}$$

And by (1), $W = W_2 - W_1$;

$$\therefore\ W = 3479.58 - 871.91 = 2607.67 \text{ lbs.};$$

or, $W = 1.164$ tons.

To find W' we have, from (2),

$$W' = 2W_2 - W_1,$$

and from (1), $2W_2 = 2W_1 + 2W$;

$$\therefore\ W' = W_1 + 2W;$$

$$\therefore\ W' = 871.91 + 5215.34 = 6087.25 \text{ lbs.}$$

or, $W' = 2.7175$ tons.

6. A vessel in the form of an inverted frustum of a cone, open at the top, is set upright and filled with water. If the vessel be gradually tilted until the bottom begins to appear above the water, find how much of the water will flow out; and how high the remainder will rise in the vessel when it is set up again.

The necessary figure for the problem, may be constructed as follows:—

Let AC be the bottom and BD the top of the vessel, A being the lowest point of the bottom, and B of the top, when the vessel is tilted until the bottom begins to appear above the water: then BC is a horizontal line. Complete the cone of which $ABCD$ is a frustum by producing BA and DC to meet in E. Draw EF perpendicular to BC produced, bisecting it in F; and EG perpendicular to BD, bisecting it in G, and AC in H.

Let $2R$ be the diameter of the top $=BD$,

$2r$ .... bottom $=AC$,

and h the altitude of the vessel $=GH$.

Now the boundary of the surface of the water will be an ellipse, of which BC is the major axis: let $BC=2a$. To find the minor axis, bisect BC in K, and through K draw a plane perpendicular to EG, intersecting it in L. This plane will intersect the cone in a circle; and, if ρ be its radius, we have

$$2\rho=R+r \text{ .... } (1).$$

Now the double ordinate of this circle which passes through K is the minor axis of the ellipse: let $2y$ be its length, and let $KL=\delta$;

$$\therefore\ y^2=\rho^2-\delta^2 \text{ .... } (2).$$

And by drawing a line through CL we see that

$$2\delta=R-r \text{ .... } (3).$$

Hence the area of the surface of the water $=\pi ay$, where

$$y^2=\frac{(R+r)^2}{4}-\frac{(R-r)^2}{4}=Rr;$$

$$\therefore\ \pi ay=\pi a\sqrt{Rr}.$$

Again the volume of the water which flows out of the vessel is the difference between the right cone EBD and the oblique cone EBC: let V be this volume,

$$\therefore\ V=\pi R^2\frac{EG}{3}-\pi a\sqrt{Rr}\,\frac{EF}{3}.$$

Now by similar triangles EOF, BOG, COH,

$$\frac{EF}{EO}=\frac{BG}{BO}=\frac{CH}{CO}=\frac{BG+CH}{BC}=\frac{R+r}{2a};$$

also $$\frac{OG}{OH}=\frac{BG}{CH}=\frac{R}{r}\,;\ \therefore\ \frac{OG+OH}{OH}=\frac{R+r}{r}\,;$$

and $$OC+OH=h,\ \therefore\ OH=\frac{hr}{R+r}.$$

Again,

$$\frac{EG}{EH}=\frac{EH+GH}{EH}=1+\frac{GH}{EH}=\frac{R}{r}\,;$$

$$\therefore\ EH=\frac{hr}{R-r}\,;$$

$$\therefore\ EO=EH+OH=\frac{hr}{R-r}+\frac{hr}{R+r}=\frac{2Rrh}{R^2-r^2}\,;$$

$$\therefore\ EF=\frac{R+r}{2a}\,\frac{2Rrh}{R^2-r^2}=\frac{Rrh}{a\,(R-r)}\,;$$

and $$EG=EH+GH=\frac{hr}{R-r}+h=\frac{Rh}{R-r}.$$

Hence $$V=\frac{\pi R^3 h}{3\,(R-r)}-\frac{Rrh}{3a\,(R-r)}\,\pi a\sqrt{Rr}$$

$$=\frac{\pi Rh}{3\,(R-r)}\,\{R^2-r\sqrt{Rr}\}$$

$$=\frac{\pi R^{\frac{1}{2}} h}{3\,(R-r)}\,\{R\sqrt{R}-r\sqrt{r}\}.$$

To find the depth of the water when the vessel is set upright; let x be the depth, and V_1 the volume;

$$\therefore\ V_1=\frac{\pi x}{3}\,(r^2+rr_1+r_1^2),\ \text{Art. 87},$$

where r_1 is the radius of the top of the inverted frustum occupied by the water;

and $$\therefore\ \frac{r_1-r}{x}=\frac{R-r}{h}\,;$$

$$\therefore\ r_1=r+\frac{x}{h}\,(R-r)\,;$$

$$\therefore\ V_1 = \frac{\pi x}{3}\left\{3r^2 + 3\,\frac{rx}{h}(R-r) + \frac{x^2}{h^2}\,(R-r)^2\right\}.$$

Now since V_1 is the volume of the water which remains in the vessel;

$$\therefore\ V_1 = \pi\,\frac{h}{3}\,(R^2 + Rr + r^2) - V.$$

Hence, by equating these two expressions for V_1, x may be determined by a cubic equation when the numerical values of R, r, and h are given.

Ex. Let $R=5$, $r=3$, $h=8$ inches;

$$\therefore\ V = \frac{\pi}{3}\,\frac{5\sqrt{5}\times 8}{2}\,(5\sqrt{5} - 3\sqrt{3})$$

$$= \frac{4\pi}{3}\,5\sqrt{5}\cdot(5\sqrt{5} - 3\sqrt{3})$$

$$= \frac{\pi}{3}\times 267.65 = 280.28 \text{ cubic inches};$$

and the volume of the vessel $= \dfrac{\pi}{3}\,h\,(R^2 + Rr + r^2)$

$$= \frac{\pi}{3}\times 8\,(25 + 15 + 9)$$

$$= \frac{\pi}{3}\times 392;$$

Therefore the proportion of the water which flows out to the whole volume $= \dfrac{267.65}{392} = \dfrac{11}{16}$, very nearly.

Again, $\quad V_1 = \dfrac{\pi}{3}\,x\left(3\times 9 + 3\times\dfrac{3}{8}\times 2x + \dfrac{4}{64}\,x^2\right).$

Also $\quad V_1 = \dfrac{\pi}{3}\,(392 - 267.65) = \dfrac{\pi}{3}\times 124.35.$

Equating these values of V_1, we have

$$27x + \frac{9}{4}\,x^2 + \frac{1}{16}\,x^3 = 124.35;$$

$$\therefore\ x^3+36x^2+432x=1989.6.$$

This equation has one real positive root and only one. It lies between 3 and 4, and may be found by approximation to be 3.49129. Therefore the water rises 3.5 inches, very nearly.

7. Find the area of the convex surface, and the volume of a cylinder, the diameter of which is 3 feet, and the altitude 21 feet.

8. A cubic foot of brass is drawn into a wire one-tenth of an inch in diameter: find its length.

9. A hollow circular cylinder of cast iron is 31.43 feet in circumference, and 9 feet 9.5 inches in diameter inside; find its thickness; and its weight, if one cubic foot of the iron weigh 441 lbs.

10. A funnel, 3 feet in diameter, and 44 feet long, is made of sheet iron. How many square feet of the iron will be required; and what will be the weight of the funnel if the iron be one-eighth of an inch in thickness, and weigh 480 lbs. to the cubic foot?

11. A well is to be dug 5 feet in diameter, and 36 feet in depth: find the quantity of earth to be excavated, and the quantity of brickwork required for a lining 9 inches in thickness.

12. Find the volume of a semi-circular arch, 42 feet span, and 54 feet deep, the thickness of the arch being 32 inches.

13. Find the quantity of brickwork in a circular arch 30 feet span, 10 feet rise, and 75 feet in length, composed of seven concentric rings of brick each 4.5 inches deep.

14. A hollow cylinder is circular inside and elliptical outside: the diameter of the circle being 4 feet, the major axis of the ellipse 5 feet, and the minor axis a mean proportional between them; find the volume if the cylinder be 29 feet in length.

15. A sewer, the transverse section of which is the oval in Art. 74, and $r=2.5$ feet, has a uniform thickness of 9 inches of brickwork: find the volume in 100 yards of the length.

16. Compare the volumes of a semi-elliptic and a cycloidal arch having the same span, rise, length, and thickness at the crown, the ellipses being similar.

17. A semi-elliptic skew arch has a clear span of 34 feet,

a rise of 12.5 feet, and a uniform thickness of 27 inches; the length, measured along the springing, is 48 feet 3 inches, and the angle of askew 47° 33′: find the volume.

18. An occupation bridge under a railway embankment has a semi-circular arch of 11 feet clear span, and 13.5 inches in thickness. The clear height of the arch above the ground is 13.5 feet, and the height of the embankment 31 feet. The breadth of the embankment at top is 23 feet, and the slopes of the sides are 1 in 1; and the faces of the arch are flush with the slopes of the embankment: find the volume of the arch.

19. The base of a circular cone is 9 inches in diameter, and its altitude 21 inches: find the surface and volume.

20. The slant height of a cone is 13 feet, and the circumference of the base 15 feet; find its surface and volume.

21. How much canvas will make a conical tent 11 feet in height, and 12 feet in diameter at the base?

22. Find the volume of an elliptic cone, the axes of the base of which are 13 and 9 inches, and the altitude 33.75 inches.

23. If the diameters of the base and top of a frustum of a cone be 9 and 6 inches respectively, and the altitude 10 inches; find the area of the convex surface, and the volume.

24. The circumference of the base of a conic frustum is 49.3 feet, the diameter of the top 12.5 feet, and the altitude 15.7 feet; find the surface and volume.

25. If a cone be bisected by a plane parallel to its base, find the surface of the frustum cut off.

26. A tent is made in the form of a conic frustum, surmounted by a cone. The diameters of the base and top of the frustum are 14 and 7 feet, its height 8 feet, and the height of the tent 12 feet: find the quantity of canvas required for it.

27. The lower portion of a haystack is an inverted conic frustum, and the upper part a cone: the greatest height is 25 feet, the greatest circumference 54 feet, the height of the frustum 15 feet, and the diameter of the base 15 feet: find its content in cubic yards.

28. A circular cone is bisected by a plane inclined to the base at an angle of 45°; find the point in the axis through which the plane passes, the dimensions of the cone being given.

29. A vessel in the form of a frustum of a cone, open at the top, is filled with water, and gradually tilted until the bottom begins to appear above the water: find the quantity of water which will flow out; and the depth of the remainder when the vessel is set upright again. Ex. Radius of top = 6 inches, radius of bottom = 10 inches, depth = 8 inches.

30. Find the inclination which must be given to the vessel in the last Problem so that half the water may run out.

31. Find the volume of a paraboloid whose altitude is 7 inches, and the diameter of its base 5.3 inches.

32. The circumference of the base of a frustum of a paraboloid is 28.3 inches, the diameter of the top 6.2 inches, and the altitude 13.4 inches: find the volume.

33. A paraboloid is trisected by two planes perpendicular to the axis: compare the altitudes of the three solids into which the paraboloid is divided.

34. A paraboloid and a cone have the same base and vertex; compare their volumes.

35. Compare the volume of a paraboloid with that of the cone which circumscribes and touches it, and has the same base.

36. Show that of all cones which circumscribe a paraboloid, that which has the least volume touches the paraboloid in a circle, the plane of which bisects the axis of the paraboloid.

37. A sphere is 36 inches in diameter: find the area of its surface in square inches, and its volume in cubic inches.

38. If there be four spheres whose diameters are as the numbers 3, 4, 5, 6; show that the volume of the greatest will be equal to the sum of the volumes of the other three.

39. Supposing the earth a sphere of 7912 miles in diameter, find the number of square miles in its surface, and the number of cubic miles in its volume.

40. The altitude of a segment of a sphere is 4.3 inches, and the diameter of its base 13.5 inches: find the convex surface, and the volume.

41. A zone of a sphere is 4 inches in thickness; the diameter of the base is 12 inches, and that of the top 9 inches: find the convex surface and the volume.

42. A hemispherical shell is filled with water, and tilted until one-half runs over: find the angle through which the axis is turned.

43. A cone and a segment of a sphere are placed base to base, their bases being equal; compare their altitudes when their volumes are equal.

44. Show how to divide a sphere into three equal portions by parallel planes.

45. Two spheres are suspended in a room between two parallel walls, with their centres in a horizontal line perpendicular to the walls. Required to place a luminous point on the same horizontal line, so that the shadows of the spheres upon the opposite walls may be equal in area.

46. Find the thickness of an 8-inch shell, if it weigh half as much as a solid ball of the same diameter.

47. Find the weight of a 13-inch iron shell, the thickness of which is 2 inches; the weight of a cubic foot of iron being 441 lbs.

48. A spherical shell weighs nineteen-twentieths of a solid sphere of the same size and material: find the diameter of the internal cavity.

49. What weight of powder will fill a shell whose diameter is 13 inches, and thickness 2 inches; if 30 cubic inches of powder weigh 1 lb.?

50. A cast iron shell, 8 inches in diameter, is filled with gunpowder and plugged with iron; and the whole weighs 75.5 lbs. Find the thickness of the shell, supposing a cubic foot of iron to weigh 441 lbs., and a cubic foot of gunpowder 57.6 lbs.

51. What length of a smooth-bore gun will be filled by 12 lbs. of powder, if the gun throw a ball of 32 lb., and $\frac{1}{49}$th of the ball's diameter be allowed for windage; a 9 lb. ball being 4 inches in diameter, and 1 lb. of powder occupying 30 cubic inches?

52. A cylindrical hole is to be drilled through a sphere, the axis of the cylinder passing through the centre of the sphere; what diameter must be given to the hole so that the sphere may be diminished in weight by one-fourth?

53. A conical hole is drilled in a sphere, the vertex of the cone being at the centre of the sphere, and its vertical angle 40°;

find how much the sphere will lose in weight, if its original weight be 77 lbs.

What must be the angle of the cone that the sphere may lose one-third of its weight?

54. If an infinite series of spheres be placed in a hollow inverted circular cone, each sphere touching the conical surface and the sphere immediately below it, and the highest touching the plane of the base of the cone; show that

$$\frac{V}{V'} = \frac{2\cos^2 a}{3 + 2\sin^4 a},$$

V being the volume of all the spheres, V' the volume of the cone, and $2a$ its vertical angle.

Determine the form of the cone so that $V' = 2V$.

55. How much of the earth's surface could a person see if he were raised $\frac{1}{n}$th part of the diameter above it? Ex. Suppose the height to be 6 miles, and the diameter of the earth 7912 miles.

56. If the nature of the earth's crust be known to an average depth of 5 miles, what proportion of its whole volume is known, supposing it a sphere of 7912 miles in diameter?

57. If the ocean cover three-quarters of the surface of the globe and its mean depth be taken at 4 miles, what proportion does its volume bear to that of the globe?

58. If the atmosphere extend to a height of 50 miles above the earth's surface, what proportion does its volume bear to that of the earth, considered a sphere of 7912 miles in diameter?

59. An ellipse, whose axes are 12 and 7 inches, generates a prolate spheroid: find its volume; and also the volume of the middle zone which has a thickness of 9 inches.

60. The true figure of the earth is an oblate spheroid, the polar axis of which is 7899.6 miles in length, and the equatorial diameter 7926.05 miles; find its volume, and the diameter of the sphere of equal volume.

Find also the radius and the positions of the two equal circles in which this sphere intersects the surface of the earth.

61. Compare the surfaces of the torrid, temperate, and frigid zones of the earth, supposing the former to extend to an angular distance of 23° 30′ from the equator, and the latter to a

distance of 23° 30′ from the pole; the earth being considered a sphere of 7912 miles in diameter.

62. Compare the volumes of the frigid, temperate, and torrid zones of the earth, cut off by planes parallel to the equator,

(1) supposing the earth a sphere of 7912 miles in diameter;

(2) supposing the earth an oblate spheroid as in 60.

63. If R and r be the radii of the exterior and interior surfaces of a spherical shell, and $2k$ the thickness of a middle zone equal in volume to half the shell; show that

$$k=\frac{1}{3}\,\frac{R^2+Rr+r^2}{R+r}.$$

64. A conical hole is made in a spherical shell, the vertex of the cone being in the centre of the shell. The outer diameter of the hole is 2.25 inches, the diameter of the shell 13 inches, and the thickness of the material 2.25 inches: what part of its weight does the shell lose?

65. The axes of an ellipsoid are 21.3, 15.2, and 12.4 inches: find its volume.

66. Find the volume of a segment of the ellipsoid in the last Example, made by a plane perpendicular to the mean axis, the height of the segment being 6.2 inches.

67. Show that the volume of an ellipsoid is two-thirds of the volume of each of the three elliptic cylinders which circumscribe it.

68. An ellipsoid, whose semi-axes are 7, 8, and 9 inches, is cut by a plane perpendicular to the mean axis, at a distance of 1.5 inches from the centre: find the volume of the segment cut off.

69. The thickness of a solid circular ring is 1.2 inches, and its outside diameter 7.1 inches; find its surface and volume.

70. Compare the weights of two solid circular rings which have the same transverse section, and one of which just passes through the other.

71. Find the internal diameter of a hollow circular ring, which weighs the same as a solid ring, of the same material and section, which just passes through it.

72. The centre of a small circle, whose radius is b, describes the involute of a circle whose radius is a, with its plane

always touching the latter: show that the volume of the solid generated in the first r revolutions is $2\pi^2 r^2 ab^2$.

73. In the last Example show that b cannot exceed πa; and that when $b=\pi a$, the solid generated will be that formed by a wire, or rope, coiled closely in one plane.

74. Show that the volume of a thin oblate spheroidal shell, contained between two concentric, similar, and similarly situated spheroids is very nearly $=4\pi abh$; where h is the thickness of the shell, and a, b the semi-axes major and minor of the generating ellipse.

75. Find the volume of the middle zone of an ellipsoidal shell contained between two similar, concentric, and similarly situated ellipsoids; the axes of the external surface of the shell and one axis of the internal surface being given, and also the length of the zone.

76. A semi-ellipsoid, made by a plane perpendicular to the greatest axis, is set upright and circumscribed by an elliptic cone which touches it in an ellipse whose plane is at a height x above the base: find the volume of this cone.

77. A dome is formed between two paraboloids of revolution which have a common axis and common focus; the diameter of the interior of the dome at the base is 100 feet, the clear height 109 feet, and the circumference of the base 261 feet: find the volume.

78. In the last Problem, if a conical hole be made near the top of the dome, with its generating line a normal to the exterior surface, and its greatest diameter 14 feet: find the volume of the dome.

79. A wire one-tenth of an inch in diameter is coiled closely upon a cylinder 12 inches in circumference, and there are 100 complete coils; determine the length of the wire.

80. A wire cable is formed of 6 wires twisted round a central one, the diameter of each being one-eighth of an inch. The central wire is straight and the others make one turn in 8 inches: find the volume of a yard of the cable.

81. A tube is generated by the motion of the centre of a circular annulus along a helix, the plane of the annulus being always perpendicular to the tangent to the helix: find the volume of one complete coil of the tube, supposing the helix to be de-

scribed upon a cylinder of 18 inches diameter, and to rise 11 inches in one revolution; the internal and external diameters of the tube being 1.5 and 2 inches respectively.

82. If a segment of a parabola, terminated both ways by a chord perpendicular to the axis, revolve round the tangent at its vertex, show that the volume of the solid generated will be one-fifth of its circumscribing cylinder.

83. Find the volume of the solid generated by the revolution of a parabolic arc round a line perpendicular to the axis, but which does not meet the curve.

Examples and Problems on Chapter IX.

We have already given examples in illustration of the "Prismoidal Formula," commencing at page 167. All the examples of the preceding Chapter may also be solved by means of this formula, and the student is recommended to solve some of them again in this manner. We add a few more examples for practice.

1. In a frustum of a semi-cubical paraboloid of revolution, the area of the transverse section at distance x from the smaller end of the solid is expressed by

$$A' + Bx + Cx^2 + Dx^3;$$

determine the values of A', B, C, and D; the length of the solid being 9 inches, and the radii of its ends 2 inches, and 5.6 inches.

Let y be the radius of the section at distance x,

$$\therefore \pi y^2 = A' + Bx + Cx^2 + Dx^3.$$

Again, if h be the length of the solid cut off from the smaller end of the frustum,

$$y^2 \propto (h+x)^3;$$

$$\therefore y^2 = m(h+x)^3,$$

where m is constant;

$$\therefore \pi y^2 = \pi m h^3 + 3\pi m h^2 x + 3\pi m h x^2 + \pi m x^3.$$

The two expressions for πy^2 are identically equal;

$$\therefore A' = \pi m h^3,\ B = 3\pi m h^2,\ C = 3\pi m h,\ D = \pi m.$$

Now when $x=0$, $y=2$; and when $x=9$, $y=5.6$.

Introducing these conditions into the equation

$$y^2=m\,(h+x)^3,$$

we have
$$2^2=mh^3,$$

$$(5.6)^2=m\,(h+9)^3;$$

$$\therefore\ \frac{h+9}{h}=\sqrt[3]{\left(\frac{5.6}{2}\right)^2}=\sqrt[3]{7.84}=1.98658;$$

$$\therefore\ h=9.1224;$$

$$\therefore\ m=\frac{4}{h^3}=.005269.$$

Hence

$$A'=4\pi;\ B=1.31544\pi;\ C=0.14421\pi;\ D=0.00527\pi.$$

Substituting these values of A', B, C, and D, in the expression for the area of the section at distance x from the smaller end of the solid, viz.

$$A'+Bx+Cx^2+Dx^3,$$

we have

$$\text{area of section}=\pi\,(4+1.31544x+.14421x^2+.00527x^3).$$

The Student should observe the rapid decrease in the value of the coefficients of the powers of x, as those powers increase.

2. Find an expression for the volume of a semi-cubical paraboloid, generated by the revolution of a semi-cubical parabola round its axis. In this curve $y^2 \propto x^3$, y being the revolving ordinate.

3. A paraboloid and a semi-cubical paraboloid have a common base and vertex; show that their volumes are as 2 : 1.

4. In a frustum of a cone, the area of the transverse section at distance x from one end of the solid is expressed by

$$A'+Bx+Cx^2;$$

find the values of A', B, and C; the length of the frustum being 10 inches, and the radii of its ends 3 inches, and 7 inches.

5. A hyperboloid is generated by the revolution of a hyperbola round it major axis; find the volume of a segment of the solid cut off by a plane perpendicular to the axis of revolution.

6. A hyperboloid is generated by the revolution of a hyperbola round its minor axis; find the volume of a zone contained between two planes perpendicular to the axis, and equidistant from the centre.

7. A hollow cone is generated by the revolution, round the major axis, of the figure contained between a hyperbola and its asymptotes: find its volume.

8. A vessel, whose interior surface has the form of a prolate spheroid, is placed with its axis vertical, and filled with a fluid to a depth h; find the depth of the fluid when the axis is horizontal.

9. A square-threaded screw, with double thread, is formed upon a solid cylinder 3 inches in diameter: the thread projects from the cylinder $\frac{5}{16}$ths of an inch, and the screw rises 3 inches in four turns: find the volume, if the screw be 9 inches in length.

10. An elliptic paraboloid is constructed with two parabolas, the area of any segment of the one being always twice that of the corresponding segment of the other: find the volume of a segment of the paraboloid.

11. Find the volume of a square groin, the base of which is 15 feet square, and the guiding curve a semicircle.

12. Find the volume of an octagonal vault, the base of which is a regular octagon of 9 feet in the side; and the guiding curve a parabola, with its plane vertical, and passing through the greatest diagonal of the base; the greatest height of the vault being 28 feet.

13. An elliptic hyperboloid is constructed with two hyperbolas which have a common major axis 14 inches in length; their minor axes being 11 inches and 8 inches: find the volume of a segment of the hyperboloid 33 inches in length; and determine the position of a plane, perpendicular to the major axis of the solid, which shall bisect this segment.

14. The surface of a solid is generated by the motion of a straight line which always passes through a fixed line, and through two ellipses whose centres are in this line and their planes perpendicular to it, their major axes being at right angles to each other. Find the volume of the solid in terms of the axes of the ellipses and the distance between their planes.

Examples and Problems on Chapter X.

For Examples of Cask gauging see Arts. 112, 113.

1. A parabolic spindle is 15 inches in length, and its greatest diameter is 7 inches: find its volume.

2. Compare the volume of a parabolic spindle with that of a prolate spheroid which has the same axis and the same mid-section.

3. A parabolic spindle is equal in volume to one-fifth of the sphere on its axis as diameter; show that its greatest diameter is equal to half its length.

4. A frustum of a parabolic spindle, one end of which passes through the axis of the generating parabola, is 33 inches in length, and its greatest and least diameters are 36 and 30 inches respectively: find its volume.

5. A parabolic spindle is placed in a cylinder half full of water, the greatest diameter of the spindle being equal to that of the interior of the cylinder; find the height of the cylinder so that the water may just rise to the top.

6. A parabolic spindle is placed in a cylinder half full of water, the length of the spindle being equal to that of the cylinder: find the diameter of the cylinder so that the water may just rise to the top.

7. A middle frustum of a parabolic spindle has its greatest and least diameters in the ratio 4 : 3, the greatest being equal to the length of the solid: find its volume in terms of its length.

8. A cask has the form of a middle frustum of a prolate spheroid, the head diameter being 24 inches, the bung diameter 32 inches, and the length 40 inches; find its capacity in imperial gallons.

9. If a cask be made in the form of a middle frustum of a parabolic spindle, the head diameter being 18 inches, the bung diameter 24 inches, and the length 30 inches; what will be its capacity in imperial gallons?

10. A cask in the form of a double frustum of two equal paraboloids has its bung diameter 36 inches, its head diameter 27 inches, and its length 45 inches; find its capacity in imperial gallons.

11. Supposing the cask in the last Example to be a double frustum of two equal cones; find its capacity.

12. Find the capacity of a cask by Hutton's formula, the bung diameter being 45 inches, the head diameter 34.2 inches, and the length 56.4 inches.

13. The dimensions of a cask being as follow: head diameter 28.6, bung diameter 37.3, length 52.4; find its capacity by each of the formulæ of Art. 113, and determine the extreme limits of the error, supposing the true capacity of the cask to lie between the greatest and least of the values given by those formulæ.

14. The bung diameter of a cask being 45 inches, the head diameter 34.2 inches, and the length 56.6 inches; and the cask being intermediate in capacity to the middle frustum of a parabolic spindle, and the double frustum of two equal paraboloids; find the greatest error in the gauging if either of those solids be taken as the true form of the cask.

15. If the bung diameter of a cask be 45 inches, the head diameter 34 inches, the mean diameter 40.5 inches, and the length 56 inches; find its capacity in imperial gallons.

Examples and Problems on Chapter XI.

EXAMPLES in illustration of the formulæ of this Chapter have already been given in Arts. 122 and 123; and other examples upon the first two formulæ of Art. 120 have been given in the 'Examples and Problems on Chapter III.' In all the practical applications of these formulæ the lengths of the equidistant ordinates are obtained by *actual measurement;* and the areas of the equidistant sections are calculated from the most convenient measurements which can be made. Hence it follows that examples in this part of the subject do not admit of much variety; and the student who can work out one or two with facility, can work out any other which he may meet with in practice. We shall therefore add only a very few examples here to those which have already been given.

1. Find, approximately, the area of a plot of ground bounded on one side by a straight line, and on the other by an irregular curve; the length of the straight line being 8.97

chains; and the lengths, in chains, of 31 equidistant ordinates (taken in order) measured from this line to the opposite boundary being as follow:

1.37, 1.83, 2.31, 2.52, 2.58, 2.47, 2.32, 2.46, 2.73, 2.91,
3.21, 2.75, 2.63, 2.01, 1.98, 1.74, 1.50, 1.59, 1.98, 2.37,
2.76, 2.93, 3.41, 3.92, 3.42, 3.00, 2.76, 2.42, 2.01, 1.75, 1.50.

2. Find, approximately, by the formulæ (2) and (3) of Art. 120, the area of a portion of a cubical parabola, contained between two ordinates whose lengths are 7 and 23 inches, and their distance 6 inches; employing 7 equidistant ordinates calculated from the equation to the curve, viz. $a^2y=x^3$; and compare the results with the exact value of the area obtained by the method of limits.

3. Two ordinates of a semi-cubical parabola are 23 and 39.5 inches in length, and their distance is 12 inches; find, approximately, by Weddle's rule, the area of the figure contained between them and the curve, from thirteen equidistant ordinates calculated from the equation to the curve, viz.

$$ay^2=x^3.$$

4. A vessel laden with a cargo floats at rest in still water, and the line of flotation is marked. Upon the removal of the cargo every part of the vessel rises 3 feet, when the line of flotation is again marked. From the known lines of the vessel the areas of the two planes of flotation and of five intermediate equidistant sections are calculated and found to be as follows, the areas being expressed in square feet:

3918, 3794, 3661, 3517, 3361, 3191, 3004:

find the weight of the cargo removed, supposing 355 cubic feet of water to weigh 10 tons.

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